Titre : Inégalités de martingales non commutatives et Applications

Mathilde Perrin

École Doctorale Louis Pasteur

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Mathilde Perrin

Inégalités de martingales non commutatives et Applications

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dirigée par Quanhua Xu

Rapporteurs : Gilles Pisier et Narcisse Randrianantoanina

Soutenue le 5 juillet 2011 devant le jury composé de :

M. Uwe Franz	Examinateur
M. Javier Parcet	Examinateur
M. Gilles Pisier	Rapporteur
M. Narcisse Randrianantoanina	Rapporteur
M. Eric Ricard	Examinateur
M. Quanhua Xu	Directeur

Laboratoire de Mathématiques de Besançon
16 route de Gray
25030 Besançon

École doctorale Louis Pasteur

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Résumé

Résumé

Cette thèse présente quelques résultats de la théorie des probabilités non commutatives, et traite en particulier des inégalités de martingales dans des algèbres de von Neumann et de leurs espaces de Hardy associés. La première partie démontre un analogue non commutatif de la décomposition de Davis faisant intervenir la fonction carrée. Les arguments classiques de temps d'arrêt ne sont plus valides dans ce cadre, et la preuve se base sur une approche duale. Le deuxième résultat important de cette partie détermine ainsi le dual de l'espace de Hardy conditionnel $\mathsf{h}_1(\mathcal{M})$. Ces résultats sont ensuite étendus au cas $1 < p < 2$. La deuxième partie transfère une décomposition atomique pour les espaces de Hardy $\mathsf{h}_1(\mathcal{M})$ et $\mathcal{H}_1(\mathcal{M})$ aux martingales non commutatives. Des résultats d'interpolation entre les espaces $\mathsf{h}_p(\mathcal{M})$ et $\mathsf{bmo}(\mathcal{M})$ sont également établis, relativement aux méthodes complexe et réelle d'interpolation. Les deux premières parties concernent des filtrations discrètes. Dans la troisième partie, on introduit des espaces de Hardy de martingales non commutatives relativement à une filtration continue. Les analogues des inégalités de Burkholder/Gundy et de Burkholder/Rosenthal sont obtenues dans ce cadre. La dualité de Fefferman-Stein ainsi que la décomposition de Davis sont également transférées avec succès à cette situation. Les preuves se basent sur des techniques d'ultraproduit et de L_p-modules. Une discussion sur une décomposition impliquant des atomes algébriques permet d'obtenir les résultats d'interpolation attendus.

Mots-clefs

Algèbres de von Neumann, espaces L_p non commutatifs, martingales non commutatives, espaces de Hardy, fonctions carrées, interpolation.

Non commutative martingale inequalities and applications

Abstract

This thesis presents some results of the theory of noncommutative probability. It deals in particular with martingale inequalities in von Neumann algebras, and their associated Hardy spaces. The first part proves a noncommutative analogue of the Davis decomposition, involving the square function. The usual arguments using stopping times in the commutative case are no longer valid in this setting, and the proof is based on a dual approach. The second main result of this part determines the dual of the conditioned Hardy space $\mathsf{h}_1(\mathcal{M})$. These results are then extended to the case $1 < p < 2$. The second part proves that an atomic decomposition for the Hardy spaces $\mathsf{h}_1(\mathcal{M})$ and $\mathcal{H}_1(\mathcal{M})$ is valid for noncommutative martingales. Interpolation results between the spaces $\mathsf{h}_p(\mathcal{M})$ and $\mathsf{bmo}(\mathcal{M})$ are also established, with respect to both complex and real interpolations. The two first parts concern discrete filtrations. In the third part, we introduce Hardy spaces of noncommutative martingales with respect to a continuous filtration. The analogues of the Burkholder/Gundy and Burkholder/Rosenthal inequalities are obtained in this setting The Fefferman-Stein duality and the Davis decomposition are also successfully transferred to this situation. The proofs are based on ultraproduct techniques and L_p-modules. A discussion about a decomposition involving algebraic atoms gives the expected interpolation results.

Keywords

Von Neumann algebras, noncommutative L_p-spaces, noncommutative martingales, Hardy spaces, square functions, interpolation.

Table des matières

Introduction

Cette thèse s'inscrit dans la théorie des probabilités et de l'intégration non commutative, qui trouve son inspiration dans la physique quantique. L'idée fondamentale sur laquelle se base cette théorie est de remplacer les fonctions par des opérateurs sur un espace de Hilbert et les mesures par des traces. Les inégalités de martingales non commutatives et leurs espaces de Hardy associés en particulier sont au coeur de ce travail de thèse. En théorie des probabilités, les interactions entre les inégalités de martingales et l'analyse harmonique sont nombreuses. Burkholder, Davis, Gundy, Doob, Meyer, Neveu et beaucoup d'autres ont développé dans ce cadre de puissants outils tels que les transformées de martingales, les fonctions maximales et les temps d'arrêt, qui jouent un rôle important dans la théorie des processus stochastiques. Des outils supplémentaires d'analyse fonctionnelle et combinatoire sont cependant nécessaires pour étendre les inégalités de martingales classiques au cadre non commutatif. Par exemple, la plupart des arguments de temps d'arrêt ne sont plus valides dans ce cadre, et la notion de fonction maximale ne peut s'étendre directement à des opérateurs dans la mesure où, en général, parler du supremum d'une suite d'opérateurs n'a aucun sens. La théorie des martingales non commutatives a connu un développement spectaculaire depuis l'article déterminant [35], et aujourd'hui de nombreuses inégalités de martingales ont été transférées avec succès au cas non commutatif. Les techniques développées dans ce domaine permettent parfois d'obtenir de nouveaux résultats dans la théorie classique, comme illustré dans [24]. Ces investigations permettent également d'enrichir les connaissances sur les C*-algèbres ou les algèbres de von Neumann qui constituent le cadre de la théorie non commutative.

Dans cette introduction, je rappellerai dans un premier temps quelques résultats bien connus de la théorie classique des martingales à temps discret. Je m'intéresserai ensuite à leurs analogues non commutatifs, en citant les inégalités de martingales non commutatives dues à Pisier, Xu, Junge, Parcet, Randrianantoanina, Musat et d'autres. Les résultats obtenus avec mes co-auteurs Bekjan, Chen, Yin et Junge et présentés dans cette thèse seront également détaillés. Ces travaux concernent essentiellement les espaces de Hardy conditionnels de martingales non commutatives. Dans la dernière partie de cette introduction, j'aborderai l'extension de toute cette théorie aux martingales non commutatives à temps continu, effectuée en collaboration avec Junge.

0.1 Théorie classique des martingales à temps discret

Les principaux objets étudiés dans ce travail sont les inégalités de martingales et leurs espaces de Hardy associés. En théorie des probabilités classiques, les espaces de Hardy de martingales sont étroitement liés aux espaces de Hardy de fonctions introduits en analyse harmonique. Nous rappelons ci-dessous quelques unes de leurs nombreuses caractérisations, qui donneront la trame de l'étude noncommutative. On se réfère au livre de Garsia [10] pour la théorie des inégalités de martingales. Considérons un espace probabilisé $(\Omega, \mathcal{F}, \mu)$

muni d'une filtration croissante $(\mathcal{F}_n)_{n\geq 0}$ de sous-σ-algèbres de $\mathcal{F}$ telle que $\mathcal{F} = \sigma\big(\cup_n \mathcal{F}_n\big)$. Soit $(\mathbb{E}_n)_{n\geq 0}$ la suite des espérances conditionnelles associées. Une martingale relativement à la filtration $(\mathcal{F}_n)_{n\geq 0}$ est une suite de variables aléatoires $(f_n)_{n\geq 0}$ dans $L_1(\Omega)$ telle que

$$\mathbb{E}_n(f_{n+1}) = f_n \quad \text{pour tout } n \geq 0. \tag{0.1.1}$$

Pour $1 \leq p \leq \infty$, on dit que la martingale f est bornée dans $L_p(\Omega)$ si $\|f\|_p = \sup_n \|f_n\|_p < \infty$. À une martingale $f = (f_n)_{n\geq 0}$ bornée dans $L_1(\Omega)$ on peut associer sa fonction carrée

$$S(f) = \Big(\sum_n |df_n|^2\Big)^{1/2},$$

où $df_n = f_n - f_{n-1}$, et sa fonction maximale

$$M(f) = \sup_n |f_n|.$$

Pour $1 \leq p < \infty$, l'espace de Hardy de martingales $H_p(\Omega)$ est défini comme l'ensemble des martingales f bornées dans $L_p(\Omega)$ telles que $S(f) \in L_p(\Omega)$. On munit cet espace de la norme

$$\|f\|_{H_p(\Omega)} = \|S(f)\|_p.$$

On introduit également l'espace

$$BMO(\Omega) = \{f \in L_2(\Omega) \ : \ \sup_n \|\mathbb{E}_n|f - f_{n-1}|^2\|_\infty < \infty\}$$

muni de la norme

$$\|f\|_{BMO(\Omega)} = \sup_n \|\mathbb{E}_n|f - f_{n-1}|^2\|_\infty^{1/2}.$$

Cette terminologie est justifiée par le fait que pour un choix approprié de $(\Omega, \mathcal{F}, \mu)$ et $(\mathcal{F}_n)_{n\geq 0}$, l'espace $H_p(\Omega)$ peut s'identifier aux espaces de Hardy classiques de la théorie des fonctions et $BMO(\Omega)$ à la classe des fonctions à oscillation moyenne bornée introduite par John et Nirenberg. Fefferman et Stein ont établi la dualité suivante entre ces deux espaces

$$H_1(\Omega)^* = BMO(\Omega). \tag{0.1.2}$$

Ce résultat de dualité va jouer un rôle fondamental dans les travaux présentés dans cette thèse. L'espace de Hardy $H_p(\Omega)$ peut aussi se caractériser à l'aide de la fonction maximale de la façon suivante. Soit f une martingale bornée dans $L_p(\Omega)$, on dit que $f \in H_p^{\max}(\Omega)$ si $M(f) \in L_p(\Omega)$. À l'aide des inégalités de Burkholder-Davis-Gundy, qui établissent que pour $p \geq 1$ et une martingale f bornée dans $L_p(\Omega)$ on a

$$\|S(f)\|_p \simeq \|M(f)\|_p, \tag{0.1.3}$$

on déduit que $H_p(\Omega) = H_p^{\max}(\Omega)$ avec des normes équivalents pour $1 \leq p < \infty$. La célèbre inégalité maximale de Doob établit que

$$\|M(f)\|_p \leq \delta_p \|f\|_p \quad \text{pour } 1 < p \leq \infty, \tag{0.1.4}$$

et les inégalités de Burkholder-Gundy démontrent que

$$\|f\|_p \simeq_{c_p} \|S(f)\|_p \quad \text{pour } 1 < p < \infty. \tag{0.1.5}$$

Ces inégalités signifient que pour $1 < p < \infty$, les espaces $H_p(\Omega)$ et $H_p^{\max}(\Omega)$ coïncident en fait avec $L_p(\Omega)$.

Les transformées de martingales constituent un outil puissant en probabilité et dans d'autres domaines de l'analyse. Par exemple, Burkholder a démontré dans [3] que les transformées de martingales sont de type $(1,1)$ faible, ce qui permet de démontrer d'autres inégalités.

Dans les travaux de Burkholder et Gundy, un certain nombre de résultats concernant la fonction carrée $S(f)$ ont aussi été obtenus pour la fonction carrée conditionnelle

$$s(f) = \Big(\sum_n \mathbb{E}_{n-1}|df_n|^2\Big)^{1/2}.$$

En effet, les inégalités de Burkholder ([4, 5]) établissent le résultat suivant

$$\|f\|_p \simeq \Big(\sum_n \|df_n\|_p^p\Big)^{1/p} + \|s(f)\|_p \quad \text{pour } 2 \leq p < \infty. \tag{0.1.6}$$

La fonction carrée conditionnelle $s(f)$ joue un rôle important dans la preuve de Davis des inégalités (0.1.3) dans le cas $p = 1$ ([7]), dans laquelle apparaît la caractérisation suivante de $H_1^{\max}(\Omega)$

$$M(f) \in L_1(\Omega) \Leftrightarrow \begin{array}{l} f \text{ se décompose en une somme de deux martingales} \\ f = g + h \text{ satisfaisant } s(g) \in L_1(\Omega) \text{ et } \sum_n |dh_n| \in L_1(\Omega). \end{array} \tag{0.1.7}$$

Cette décomposition est connue sous le nom de décomposition de Davis. Si on note $h_1(\Omega)$ l'espace des martingales dans $L_1(\Omega)$ qui admettent une telle décomposition, les résultats précédents entraînent

$$H_1(\Omega) = h_1(\Omega) = H_1^{\max}(\Omega). \tag{0.1.8}$$

Rappelons que le dual de l'espace $h_1(\Omega)$ est bien connu, et décrit comme l'espace appelé petit *bmo* (voir par exemple [37]).

D'autres décompositions jouent un rôle important dans la théorie des martingales. La décomposition de Gundy ([12]) permet par exemple d'établir des résultats de bornitude $(1,1)$ faible pour les fonctions carrées et maximales, qui permettent à leur tour de retrouver certaines inégalités citées précédemment.

La décomposition atomique constitue un outil puissant pour démontrer des résultats de dualité, d'interpolation et certaines inégalités fondamentales à la fois en théorie des martingales et en analyse harmonique. Elle a été introduite par Coifman [6] en analyse harmonique et par Herz [17] en théorie des martingales. Les atomes pour les martingales sont traditionnellement définis à l'aide de temps d'arrêt. Dans l'optique d'étendre cette décomposition au cadre non commutatif, nous mentionnons une approche différente dans laquelle la définition d'atome ne fait pas intervenir la notion de temps d'arrêt. En effet, comme évoqué précédemment, le concept de temps d'arrêt n'est pour l'instant pas clairement défini dans le cas non commutatif. On dit qu'une fonction $\mathcal{F}$-mesurable $a \in L_2(\Omega)$ est un atome s'il existe $n \in \mathbb{N}$ et $A \in \mathcal{F}_n$ tels que

(i) $\mathbb{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Cette notion a été introduite par Weisz dans [50] sous le nom d'atomes simples, puis étudiée dans [49, 50]. Sous une forme déguisée dans la preuve du Théorème A_∞ de [17], Herz décompose en atomes l'espace des martingales prévisibles $\mathcal{P}_1(\Omega)$. On rappelle qu'une martingale $f = (f_n)_{n\geq 0}$ est dite prévisible dans L_1 s'il existe une suite adaptée $(\lambda_n)_{n\geq -1}$ de fonctions croissantes et positives telle que $|f_n| \leq \lambda_{n-1}$ pour tout $n \geq 0$ et $\sup_n \lambda_n \in L_1(\Omega)$. L'espace $\mathcal{P}_1(\Omega)$ coïncidant avec l'espace de Hardy $H_1(\Omega)$ pour les martingales régulières, cela donne une décomposition atomique de $H_1(\Omega)$ dans le cas régulier. Cette décomposition a été étendue au cas général par Weisz dans [49] pour les martingales f dans $L_1(\Omega)$ telles que la fonction carrée conditionnelle $s(f) \in L_1(\Omega)$ (au lieu de la fonction carrée $S(f)$).

0.2 Théorie non commutative des martingales à temps discret

Nous examinons maintenant la théorie évoquée précédemment dans un cadre non commutatif, c'est-à-dire lorsqu'on remplace les fonctions par des opérateurs sur un espace de Hilbert. Après avoir décrit la construction des espaces de Hardy dans ce cadre et cité certains résultats majeurs de la théorie des martingales non commutatives, je détaillerai les trois points étudiés en particulier dans cette thèse dans le cas d'une filtration discrète, qui enrichissent la connaissance de ces espaces de Hardy en étudiant plus particulièrement leurs versions conditionnelles. Il s'agit de la décomposition de Davis, de la décomposition atomique et de l'interpolation des espaces de Hardy de martingales non commutatives.

Le cadre de la théorie des martingales non commutatives est donné par une algèbre de von Neumann $\mathcal{M}$, c'est-à-dire une sous-algèbre involutive unitale préfaiblement fermée dans l'espace $B(\mathcal{H})$ de opérateurs bornés sur un espace de Hilbert $\mathcal{H}$. Par souci de simplicité on supposera que $\mathcal{M}$ est finie, c'est-à-dire qu'il existe une trace τ normale, fidèle et normalisée. Ainsi $(\mathcal{M}, \tau)$ joue le rôle de l'espace probabilisé $(\Omega, \mathcal{F}, \mu)$. Le rôle des espaces $L_p(\Omega)$ est alors tenu par les espaces L_p non commutatifs $L_p(\mathcal{M}, \tau)$ (voir [36]), dont la norme pour $1 \leq p < \infty$ est simplement définie par

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p} \quad \text{pour } x \in L_p(\mathcal{M}),$$

où $|x| = (x^*x)^{1/2}$ désigne le module de x. Pour $p = \infty$, $L_\infty(\mathcal{M})$ désigne $\mathcal{M}$ muni de sa norme en tant qu'algèbre de von Neumann. On considère également une filtration croissante $(\mathcal{M}_n)_{n\geq 0}$ de sous-algèbres de $\mathcal{M}$, et la suite $(\mathcal{E}_n)_{n\geq 0}$ des espérances conditionnelles associées. Armés de ce dictionnaire, on peut facilement définir une martingale non commutative en traduisant simplement la condition (0.1.1) dans ce cadre. Ainsi on dira qu'une suite $(x_n)_{n\geq 0}$ d'opérateurs dans $L_1(\mathcal{M})$ est une martingale non commutative relativement à $(\mathcal{M}_n)_{n\geq 0}$ si

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{pour tout } n \geq 0.$$

Comme évoqué ci-dessus, les notions de fonction maximale et de supremum d'une suite d'opérateurs n'ont pas de sens dans ce cadre. Nous nous intéressons donc essentiellement à l'espace de Hardy quadratique défini à partir de la fonction carrée.

Il existe plusieurs manières de considérer le carré d'un opérateur x, par exemple le carré de son module $|x|^2 = x^*x$, et le carré du module de son adjoint $|x^*|^2 = xx^*$. Pisier et Xu ont ainsi naturellement introduit dans [35] deux fonctions carrées, colonne et ligne respectivement, qui définissent deux versions colonne et ligne de l'espace de Hardy de martingales non commutatives

$$\|x\|_{\mathcal{H}_p^c(\mathcal{M})} = \Big\|\Big(\sum_n |dx_n|^2\Big)^{1/2}\Big\|_p \quad \text{et} \quad \|x\|_{\mathcal{H}_p^r(\mathcal{M})} = \Big\|\Big(\sum_n |dx_n^*|^2\Big)^{1/2}\Big\|_p,$$

où $dx_n = x_n - x_{n-1}$ désigne la suite des différences de la martingale $x = (x_n)_n$. La version non commutative des inégalités de Burkholder-Gundy (0.1.5) démontrée dans [35] s'énonce alors de la manière suivante

$$\|x\|_p \simeq_{c_p} \begin{cases} \max\Big(\Big\|\Big(\sum_n |dx_n|^2\Big)^{1/2}\Big\|_p, \Big\|\Big(\sum_n |dx_n^*|^2\Big)^{1/2}\Big\|_p\Big) & \text{si } 2 \leq p < \infty \\ \inf\Big(\Big\|\Big(\sum_n |dy_n|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n |dz_n^*|^2\Big)^{1/2}\Big\|_p\Big) & \text{si } 1 < p < 2 \end{cases}, \quad (0.2.1)$$

où l'infimum est pris sur toutes les décompositions $dx_n = dy_n + dz_n$ de dx_n en somme de deux suites de différences de martingales associées à la même filtration. Cela retraduit le phénomène, découvert par Lust-Piquard et Pisier ([29, 30]) en établissant la version non commutative des inégalités de Khintchine, que les inégalités de martingales sont de natures différentes suivant que p est supérieur ou inférieur à 2. L'espace de Hardy $\mathcal{H}_p(\mathcal{M})$ est alors défini de la manière suivante

$$\mathcal{H}_p(\mathcal{M}) = \begin{cases} \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}) & \text{si } 1 \leq p < 2 \\ \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}) & \text{si } 2 \leq p < \infty \end{cases}.$$

Ainsi (0.2.1) signifie que

$$\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M}) \quad \text{avec des normes équivalentes pour } 1 < p < \infty. \quad (0.2.2)$$

La bornitude $(1,1)$ faible des transformées de martingales a également été établie dans le cadre non commutatif par Randrianantoanina dans [39], et permet en particulier de redémontrer (0.2.1) avec une meilleure constante.

En appendice de [35], Pisier et Xu ont décrit le dual de l'espace $\mathcal{H}_1(\mathcal{M})$ comme un espace de type BMO, et ont ainsi établi la version non commutative de la dualité de Fefferman-Stein (0.1.2)

$$\mathcal{H}_1(\mathcal{M})^* = \mathcal{BMO}(\mathcal{M}). \quad (0.2.3)$$

Junge et Xu ont ensuite étendu cette dualité au cas $1 \leq p < 2$ dans [24]. On trouve d'autres caractérisations de l'espace $\mathcal{BMO}(\mathcal{M})$ dans la version non commutative du Théorème de John-Nirenberg démontrée dans [22].

Concernant la fonction maximale, en s'inspirant des espaces L_p non commutatifs à valeurs vectorielles introduits par Pisier ([34]), Junge a traduit la notion de norme L_p de la fonction maximale, et a obtenu la version non commutative de l'inégalité maximale de Doob (0.1.4)

$$\|{\sup_n}^+ |\mathcal{E}_n(x)|\|_p \leq \delta_p \|x\|_p \quad \text{pour } 1 < p \leq \infty.$$

Il est important de noter qu'ici, $\|{\sup_n}^+|\mathcal{E}_n(x)|\|_p$ n'est qu'une notation, car $\sup_n |\mathcal{E}_n(x)|$ n'a pas de sens dans le cadre non commutatif. On peut ainsi définir à l'aide de cette norme l'analogue non commutatif de l'espace de Hardy maximal, noté $\mathcal{H}_p^{\max}(\mathcal{M})$. Cependant, il a été démontré dans [25] que les espaces $\mathcal{H}_1(\mathcal{M})$ et $\mathcal{H}_1^{\max}(\mathcal{M})$ ne coïncident pas en général. Plus précisément, $\mathcal{H}_1(\mathcal{M}) \not\subset \mathcal{H}_1^{\max}(\mathcal{M})$. La validité de l'inclusion inverse constitue à ce jour encore une question ouverte.

Dans l'article [24], Junge et Xu ont étendu (0.2.1) au cas non tracial, et ont démontré d'autres inégalités de martingales non commutatives, en particulier l'analogue des inégalités de Burkholder (0.1.6) pour $2 \leq p < \infty$

$$\|x\|_p \simeq \Big(\sum_n \|dx_n\|_p^p\Big)^{1/p} + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|dx_n|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|dx_n^*|^2\Big)^{1/2}\Big\|_p. \quad (0.2.4)$$

Dans le même esprit que (0.2.1), ces inégalités s'étendent au cas $1 < p < 2$ de la manière suivante

$$\|x\|_p \simeq \inf \Big(\Big(\sum_n \|dx_n^d\|_p^p \Big)^{1/p} + \Big\| \Big(\sum_n \mathcal{E}_{n-1}|dx_n^c|^2 \Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_n \mathcal{E}_{n-1}|(dx_n^r)^*|^2 \Big)^{1/2} \Big\|_p \Big), \tag{0.2.5}$$

où l'infimum est pris sur toutes les décompositions $dx_n = dx_n^d + dx_n^c + dx_n^r$ de dx_n en somme de trois suites de différences de martingales associées à la même filtration. Junge et Xu démontrent ainsi un nouveau résultat dans la théorie classique, et une version $(1,1)$ faible de ce nouveau résultat en probabilités commutatives a été obtenu par Parcet dans [32]. En introduisant les fonctions carrées conditionnelles colonne et ligne, les espaces de Hardy conditionnels

$$\|x\|_{\mathsf{h}_p^c(\mathcal{M})} = \Big\| \Big(\sum_n \mathcal{E}_{n-1}|dx_n|^2 \Big)^{1/2} \Big\|_p \quad , \quad \|x\|_{\mathsf{h}_p^r(\mathcal{M})} = \Big\| \Big(\sum_n \mathcal{E}_{n-1}|dx_n^*|^2 \Big)^{1/2} \Big\|_p$$

et l'espace de Hardy diagonal

$$\|x\|_{\mathsf{h}_p^d(\mathcal{M})} = \Big(\sum_n \|dx_n\|_p^p \Big)^{1/p},$$

les inégalités (0.2.4) et (0.2.5) se reformulent

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \quad \text{avec des normes équivalentes pour } 1 < p < \infty. \tag{0.2.6}$$

Ici on définit l'espace de Hardy conditionnel $\mathsf{h}_p(\mathcal{M})$ de la même manière que l'espace de Hardy $\mathcal{H}_p(\mathcal{M})$, en différenciant les cas $p < 2$ et $p \geq 2$:

$$\mathsf{h}_p(\mathcal{M}) = \begin{cases} \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}) & \text{si } 1 \leq p < 2 \\ \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) & \text{si } 2 \leq p < \infty \end{cases}.$$

Dans [40, 41], Randrianantoanina établit les versions $(1,1)$ faibles des inégalités (0.2.1) et (0.2.4), (0.2.5), en construisant des décompositions explicites de martingales. Dans le cas classique, ces décompositions sont basées sur des temps d'arrêt, et Randrianantoanina s'appuie sur les projections de Cuculescu pour décomposer des martingales non commutatives. Ces projections constituent un outil présentant certaines propriétés communes avec les temps d'arrêt qui sont suffisantes pour obtenir des estimations $(1,1)$ faible. Par des techniques d'interpolation réelle, Randrianantoanina en déduit certains ordres maximaux des meilleures constantes dans les inégalités (0.2.1) et (0.2.4), (0.2.5). En exploitant des techniques de décomposition similaires, Parcet et Randrianantoanina ont construit une décomposition de Gundy non commutative dans [33], qui, comme dans le cas classique, permet de redémontrer des inégalités de martingales non commutatives.

0.2.1 La décomposition de Davis

En combinant (0.2.2) avec (0.2.6), on obtient que $\mathcal{H}_p(\mathcal{M}) = \mathsf{h}_p(\mathcal{M})$ pour $1 < p < \infty$. Le résultat principal du chapitre 1 montre que cette égalité est encore vraie dans le cas $p = 1$, c'est-à-dire que la première égalité de (0.1.8) se transfère avec succès au cas non commutatif. Dans les résultats rappelés précédemment, seules des inégalités $(1,1)$ faible sont obtenues ([40, 41]), et les techniques d'interpolation ne permettent pas d'en déduire des résultats pour $p = 1$, mais uniquement pour $1 < p < \infty$. Le chapitre 1 répond positivement à une question posée dans [41], et peut être considéré comme un analogue non commutatif de la décomposition de Davis (0.1.7) faisant intervenir la fonction carrée au lieu de la fonction maximale. Cependant, la décomposition présentée dans le chapitre 1 n'est pas explicite mais démontrée par une approche duale. L'espace dual de $\mathcal{H}_1(\mathcal{M})$ est connu et donné par la dualité de Fefferman-Stein (0.2.3), le chaînon manquant était la description de l'espace dual de $\mathsf{h}_1(\mathcal{M})$ comme un espace de type BMO, analogue de l'espace petit *bmo*. On introduit dans le chapitre 1 les espaces *bmo* non commutatifs suivants : l'espace bmo colonne

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ \sup_n \|\mathcal{E}_n|x - x_n|^2\|_\infty < \infty\}$$

muni de la norme

$$\|x\|_{\mathsf{bmo}^c(\mathcal{M})} = \max\big(\|\mathcal{E}_0(x)\|_\infty \ , \ \sup_n \|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\big),$$

l'espace bmo ligne

$$\mathsf{bmo}^r(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ x^* \in \mathsf{bmo}^c(\mathcal{M})\}$$

muni de la norme

$$\|x\|_{\mathsf{bmo}^r(\mathcal{M})} = \|x^*\|_{\mathsf{bmo}^c(\mathcal{M})},$$

et l'espace bmo diagonal

$$\mathsf{bmo}^d(\mathcal{M}) = \{\text{différences de martingales dans } \ell_\infty(L_\infty(\mathcal{M}))\}$$

muni de la norme

$$\|x\|_{\mathsf{bmo}^d(\mathcal{M})} = \sup_n \|dx_n\|_\infty.$$

Dans l'esprit de la dualité classique présentée dans [37], on démontre le résultat crucial suivant.

Théorème 0.2.1. *On a* $\mathsf{h}_1(\mathcal{M})^* = \mathsf{bmo}(\mathcal{M})$ *avec des normes équivalentes, où*

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^d(\mathcal{M}) \cap \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}).$$

L'établissement de cet analogue conditionnel de la dualité de Fefferman-Stein enrichit notre connaissance de l'espace $\mathsf{h}_1(\mathcal{M})$, et permet désormais d'adopter une approche duale dans certaines questions. En particulier, en observant que les espaces $\mathcal{BMO}(\mathcal{M})$ et $\mathsf{bmo}(\mathcal{M})$ coïncident, on démontre alors le résultat principal du chapitre 1.

Théorème 0.2.2. *On a* $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})$ *avec des normes équivalentes.*

Dans le même chapitre, ces résultats sont étendus au cas $1 < p < 2$. Par une méthode similaire, on décrit le dual de l'espace $\mathsf{h}_p(\mathcal{M})$ comme l'espace $L_{p'}\mathsf{mo}(\mathcal{M})$ pour $\frac{1}{p} + \frac{1}{p'} = 1$, défini de la même manière que bmo. Plus précisément, pour $2 < q \leq \infty$ on introduit les espaces

$$L_q^c\mathsf{mo}(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ \|{\sup_n}^+ \mathcal{E}_n|x - x_n|^2\|_{q/2} < \infty\}$$

muni de la norme

$$\|x\|_{L_q^c\mathsf{mo}(\mathcal{M})} = \max\big(\|\mathcal{E}_0(x)\|_q \ , \ \|{\sup_n}^+ \mathcal{E}_n|x - x_n|^2\|_{q/2}^{1/2}\big),$$

$L_q^r\mathsf{mo}(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ x^* \in L_q^c\mathsf{mo}(\mathcal{M})\}$ et

$$L_q\mathsf{mo}(\mathcal{M}) = \mathsf{h}_q^d(\mathcal{M}) \cap L_q^c\mathsf{mo}(\mathcal{M}) \cap L_q^r\mathsf{mo}(\mathcal{M}).$$

La comparaison des espaces duaux nous permet alors d'améliorer les estimations d'une constante dans l'équivalence des normes $\mathcal{H}_p(\mathcal{M})$ et $\mathsf{h}_p(\mathcal{M})$ pour $1 < p < 2$ données dans [41]. Plus précisément, Randrianantoanina a obtenu l'estimation $\kappa_p = O((p-1)^{-1})$ lorsque $p \to 1$, où $\|x\|_{\mathsf{h}_p(\mathcal{M})} \leq \kappa_p \|x\|_{\mathcal{H}_p(\mathcal{M})}$, et notre approche montre que la constante κ_p reste bornée quand $p \to 1$. Cette approche duale permet également d'affiner les inégalités (0.2.6), en séparant les espaces ligne et colonne de la manière suivante

$$\mathcal{H}_p^c(\mathcal{M}) = \begin{cases} \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) & \text{pour} \quad 1 \leq p < 2 \\ \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) & \text{pour} \quad 2 \leq p < \infty \end{cases} . \tag{0.2.7}$$

Ces inégalités décrivent plus précisément les relations entre les espaces de Hardy colonne. De manière indépendante et au même moment, Junge et Mei ont obtenu les mêmes résultats, en décrivant le dual des espaces $\mathsf{h}_p(\mathcal{M})$ pour $1 \leq p < 2$ à l'aide d'une technique différente. Cependant la preuve présentée ici aboutit à une meilleure constante.

La décomposition de Davis joue un rôle fondamental dans l'étude des inégalités de Burkholder dans le chapitre 3, qui traite de la théorie des espaces de Hardy de martingales non commutatives à temps continu. Deux autres approches de la décomposition (0.2.7) pour $1 \leq p < 2$ sont aussi introduites dans le chapitre 3, dans le but de les transférer au cas d'une filtration continue. D'une part, en examinant de plus près les espaces duaux, une version plus forte de cette décomposition est démontrée, en remplaçant l'espace diagonal $\mathsf{h}_p^d(\mathcal{M})$ par un espace plus petit appelé $\mathsf{h}_p^{1_c}(\mathcal{M})$. On obtient ainsi une décomposition plus proche de la décomposition de Davis classique (0.1.7). D'autre part, en se basant sur une adaptation d'une décomposition de Randrianantoanina ([41]), nous discutons une autre variante de cette décomposition dans le cas $1 < p < 2$.

0.2.2 La décomposition atomique

La description du dual de $\mathsf{h}_1(\mathcal{M})$ établie dans le chapitre 1 joue un rôle fondamental dans la version non commutative de la décomposition atomique présentée dans le chapitre 2, en permettant d'aborder cette question par une approche duale. Une notion d'atome non commutatif est d'abord introduite. Plus précisément, dans l'esprit de la théorie des martingales non commutatives nous définissons deux types d'atomes, une version colonne et une version ligne. Ces notions sont une traduction directe de la définition d'atome simple rappelée dans la Section 0.1, en considérant respectivement les supports à droite et à gauche d'un opérateur. On dira qu'un opérateur $a \in L_2(\mathcal{M})$ est un $(1,2)_c$-atome s'il existe $n \in \mathbb{N}$ et une projection e dans $\mathcal{M}_n$ tels que

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{-1/2}$.

Le résultat principal de la première partie du chapitre 2 établit que l'on peut décomposer l'espace de Hardy conditionnel à l'aide de ces atomes. Plus précisément, si on note $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ (respectivement $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$) l'espace atomique dont la boule unité est constituée de l'enveloppe absolument convexe de $B_{L_1(\mathcal{M}_0)}$ et des $(1,2)_c$-atomes (resp. $(1,2)_r$-atomes), on démontre le résultat suivant.

Théorème 0.2.3. *On a $\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$ avec des normes équivalentes, où*

$$\mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_1^d(\mathcal{M}) + \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}).$$

Comme pour la décomposition de Davis évoquée précédemment, le Théorème 0.2.3 est également démontré par une approche duale, similaire à celle développée dans le chapitre 1 pour démontrer le Théorème 0.2.2. L'idée est de décrire le dual de l'espace atomique $\mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$ comme un espace de Lipschitz non commutatif, puis de le comparer au dual de l'espace $\mathsf{h}_1(\mathcal{M})$ que nous connaissons désormais comme l'espace $\mathsf{bmo}(\mathcal{M})$. Cette méthode ne donne pas de décomposition explicite, mais en démontre l'existence. La décomposition de Davis obtenue dans le Théorème 0.2.2 permet d'en déduire une décomposition atomique de l'espace $\mathcal{H}_1(\mathcal{M})$.

Dans le chapitre 3, nous discutons un autre type d'atomes, appelés atomes algébriques. Cette décomposition concerne cette fois le cas $1 \leq p < 2$. On dit qu'un opérateur $x \in L_2(\mathcal{M})$ est un $\mathsf{h}_p^c(\mathcal{M})$-atome algébrique si x se décompose sous la forme

$$x = \sum_n b_n a_n,$$

avec

(i) $\mathcal{E}_n(b_n) = 0$ pour tout $n \geq 0$;

(ii) $a_n \in L_2(\mathcal{M}_n)$ pour tout $n \geq 0$;

(iii) $\sum_n \|b_n\|_2^2 \leq 1$ et $\Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_q \leq 1$, où $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$.

Ces atomes, moins "élémentaires" que les précédents, présentent cependant la particularité de constituer un ensemble absolument convexe, et sont normants pour l'espace $L_{p'}^c\mathsf{mo}$ lorsque p' est l'indice conjugué de p. Par dualité, on démontre ainsi dans le chapitre 3 que l'espace dont la boule unité est constituée de l'enveloppe absolument convexe des $\mathsf{h}_p^c(\mathcal{M})$-atomes algébriques constitue un sous-espace dense de $\mathsf{h}_p^c(\mathcal{M})$. Plus généralement, par la même approche on décompose des espaces plus grands, appelés espaces L_p colonnes conditionnels, à l'aide d'atomes agébriques de ce type. Ces nouvelles décompositions nous fournissent des outils supplémentaires pour étudier les espaces de Hardy, et sont immédiatement appliquées dans le chapitre 3 pour obtenir des résultats d'interpolation.

0.2.3 L'interpolation des espaces de Hardy

Les premiers résultats d'interpolation des espaces de Hardy $\mathcal{H}_p(\mathcal{M})$ sont apparus dans l'article de Musat [31], dans lequel elle démontre pour la méthode complexe d'interpolation

$$(\mathcal{BMO}(\mathcal{M}), \mathcal{H}_1(\mathcal{M}))_{1/p} = \mathcal{H}_p(\mathcal{M}) \quad \text{pour } 1 < p < \infty. \tag{0.2.8}$$

Nous étendons ce résultat au cas des espaces de Hardy conditionnels $\mathsf{h}_p(\mathcal{M})$ dans le chapitre 2. La méthode utilisée est plus simple et élémentaire que les arguments développés par Musat, et permet également de retrouver (0.2.8). Il semble que, même dans le cas classique, cette méthode soit plus simple que les approches connues de l'interpolation des espaces de Hardy de martingales. L'idée principale est inspirée par une quasi norme équivalente pour $h_p(\Omega)$, $0 < p \leq 2$, introduite par Herz [18] dans le cas classique. En adaptant cette quasi norme au cadre non commutatif, nous obtenons une nouvelle caractérisation de l'espace $\mathsf{h}_p(\mathcal{M})$ qui s'avère utile pour l'interpolation.

Théorème 0.2.4. *Soit* $1 < p < \infty$. *Alors*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{1/p} = \mathsf{h}_p(\mathcal{M}) \quad \textit{avec des normes équivalentes.}$$

La caractérisation de $\mathsf{h}_p(\mathcal{M})$ à l'aide de la quasi norme de Herz permet de retrouver la dualité de Fefferman-Stein obtenue dans le chapitre 1 avec de meilleures constantes dans l'équivalence des normes $\mathsf{h}_p(\mathcal{M})^*$ et $L_{p'}\mathsf{mo}(\mathcal{M})$ pour $1 \leq p < 2$ et $\frac{1}{p} + \frac{1}{p'} = 1$. Pour $0 < p < 1$, cela ouvre une discussion sur l'espace dual de $\mathsf{h}_p(\mathcal{M})$.

Dans le chapitre 3, nous développons une autre approche de l'interpolation des espaces de Hardy. L'idée est de complémenter les espaces de Hardy dans des espaces plus grands qui forment une échelle d'interpolation. Il est bien connu que grâce à la projection de Stein on peut complémenter les espaces de Hardy dans les espaces L_p colonnes pour $1 < p < \infty$, mais cela n'inclut pas le cas $p = 1$. On introduit alors des espaces intermédiaires, les espaces L_p colonnes conditionnels, dans lesquels les espaces de Hardy sont complémentés pour $1 \leq p < \infty$. La preuve de cette complémentation se base sur une décomposition des espaces L_p colonnes conditionnels semblable à la décomposition de Davis discutée dans le chapitre 1, et démontrée toujours par une approche duale. En s'appuyant sur la décomposition de ces espaces en atomes algébriques, on démontre qu'ils s'interpolent.

0.3 Théorie non commutative des martingales à temps continu

Dans le troisième et dernier chapitre de cette thèse, nous considérons une filtration continue et étudions certains résultats cités précédemment dans ce cadre. En théorie commutative, les théories des martingales à temps continu et des intégrales stochastiques sont bien développées et connaissent de nombreuses applications. Dans le cas non commutatif, il existe une théorie du calcul stochastique quantique, qui est jusqu'à aujourd'hui seulement au niveau algébrique. Dans la ligne des investigations des martingales non commutatives détaillées ci-dessus, le chapitre 3 se propose d'étudier les martingales non commutatives à temps continu dans les cadre des algèbres de von Neumann finies. L'extension des résultats du cas discret au cas continu présente de nombreuses difficultés, et la théorie développée dans le chapitre 3 nécessite de puissants outils d'analyse fonctionnelle tels que les ultraproduits ou les L_p-modules pour les contourner. Le but à long terme de ce projet mené en collaboration avec Junge est de développer une théorie satisfaisante pour les semi martingales, incluant l'étude de la convergence des intégrales stochastiques. Dans une algèbre de von Neumann, la notion de trajectoire d'un processus d'opérateurs n'a pas de sens, et il est donc impossible de construire les intégrales stochastiques trajectoire par trajectoire, comme Dellacherie et Meyer le font dans [8]. Il est cependant bien connu que la convergence des intégrales stochastiques est étroitement liée à l'existence du crochet de variation quadratique $[\cdot, \cdot]$ via la formule

$$f_t g_t = \int^t f_{s^-} dg_s + \int^t g_{s^-} df_s + [f, g]_t.$$

Le crochet de variation quadratique peut être vu comme la limite en probabilité de la fonction carrée dyadique suivante

$$[f, g]_t = f_0 g_0 + \lim_{n\to\infty} \sum_{0 \leq k < 2^n} (f_{t\frac{k+1}{2^n}} - f_{t\frac{k}{2^n}})(g_{t\frac{k+1}{2^n}} - g_{t\frac{k}{2^n}}).$$

C'est pourquoi notre approche consiste à étudier dans un premier temps ce crochet de variation quadratique, et de traiter ensuite les intégrales stochastiques dans un prochain travail qui se basera sur la théorie développée dans le chapitre 3. En probabilités classiques, l'espace de Hardy de martingales à temps continu est défini par la norme

$$\|f\|_{H_p(\Omega)} = \|[f, f]\|_{p/2}^{1/2}.$$

Le chapitre 3 étudie l'analogue non commutatif de cet espace $H_p(\Omega)$ associé à une filtration continue.

On se place maintenant dans le même cadre que dans la Section 0.2, c'est-à-dire que l'on considère un algèbre de von Neumann finie $(\mathcal{M},\tau)$, munie cette fois d'une filtration continue $(\mathcal{M}_t)_{t\geq 0}$ de sous-algèbres de $\mathcal{M}$. Pour simplifier nous supposerons que l'ensemble continu des paramètres est donné par l'intervalle $[0,1]$. Dans l'esprit de la théorie des martingales non commutatives, on aimerait définir le crochet $[x,x]$ pour une martingale x, et poser

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]\|_{p/2}^{1/2} \quad \text{et} \quad \|x\|_{\widehat{\mathcal{H}}_p^r} = \|[x^*,x^*]\|_{p/2}^{1/2}$$

et démontrer l'analogue de (0.2.2) pour cette définition. Le candidat pour ce crochet non commutatif est défini par une approche d'analyse non standard. Pour une partition finie $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ de l'intervalle $[0,1]$ et $x\in\mathcal{M}$, on considère le crochet fini

$$[x,x]_\sigma = \sum_{t\in\sigma} |d_t^\sigma(x)|^2,$$

où $d_t^\sigma(x) = \mathcal{E}_t(x) - \mathcal{E}_{t^-}(x)$ et pour $t=t_j$ $(j=1,\cdots,n)$, $t^- = t_{j-1}$ désigne son prédécesseur dans la partition σ. Par convention on pose $d_0^\sigma(x) = \mathcal{E}_0(x)$. Alors pour $p>2$, (0.2.2) donne l'estimation $\|[x,x]_\sigma]\|_{p/2}^{1/2} \leq c_p\|x\|_p$. Si on considère un ultrafiltre $\mathcal{U}$ raffinant l'ensemble des partitions finies de $[0,1]$, on peut alors définir

$$[x,x]_{\mathcal{U}} = w\text{-}\lim_{\sigma,\mathcal{U}} [x,x]_\sigma,$$

où la limite faible est prise dans l'espace réflexif $L_{p/2}(\mathcal{M})$. En analyse non standard, la limite faible correspond à la partie standard, et il est connu que cette approche coïncide avec la définition classique du crochet pour des martingales commutatives. Cependant, la norme est seulement semi-continue par rapport à la topologie faible, et on ne peut pas obtenir les inégalités de Burkholder-Gundy pour une filtration continue comme une simple conséquence de la théorie discrète des espaces de Hardy. Néanmoins, en se basant sur l'observation cruciale que les normes $L_{p/2}$ des crochets discrets $[x,x]_\sigma$ sont monotones en σ à une constante près, on peut démontrer le résultat suivant.

Théorème 0.3.1. *Soient* $1\leq p<\infty$ *et* $x\in\mathcal{M}$. *Alors*

$$\|[x,x]_{\mathcal{U}}\|_{p/2} \simeq \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2} \simeq \begin{cases} \sup_\sigma \|[x,x]_\sigma\|_{p/2} & \text{pour} \quad 1\leq p<2 \\ \inf_\sigma \|[x,x]_\sigma\|_{p/2} & \text{pour} \quad 2\leq p<\infty \end{cases}.$$

En particulier, cela implique que la norme $L_{p/2}$ du crochet $[x,x]_{\mathcal{U}}$ ne dépend pas du choix de l'ultrafiltre $\mathcal{U}$, à norme équivalente près. L'indépendance du crochet $[x,x]_{\mathcal{U}}$ lui-même de l'ultrafiltre $\mathcal{U}$ sera discutée dans un prochain travail. Ainsi pour $1\leq p<\infty$ et $x\in\mathcal{M}$ on définit les normes

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]_{\mathcal{U}}\|_{p/2}^{1/2} \quad \text{et} \quad \|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2}^{1/2} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

On note $\widehat{\mathcal{H}}_p^c$ et $\mathcal{H}_p^c$ respectivement les complétions correspondantes. En fait ces deux procédures définissent le même espace :

$$\widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c \quad \text{avec des normes équivalentes pour} \quad 1\leq p<\infty.$$

La preuve de ce résultat est basée sur le Théorème 0.3.1, et suit deux approches différentes suivant les valeurs de p. Pour $2\leq p<\infty$, on utilise des arguments de complémentation et le cas $1 \leq p < 2$ est obtenu par dualité, en injectant $\widehat{\mathcal{H}}_p^c$ dans un gros espace ultraproduit afin d'étudier son espace dual. Nous obtenons de cette manière un bon candidat pour l'espace de Hardy de martingales non commutatives relativement à la filtration continue $(\mathcal{M}_t)_{0\leq t\leq 1}$. Nous désirons maintenant établir pour cet espace l'analogue de certains résultats présentés dans la Section 0.2 précédente. Pour cela, on travaille de préférence avec la définition de $\mathcal{H}_p^c$, qui s'avère plus maniable. En particulier, il est possible de considérer $\mathcal{H}_p^c$ comme un sous-espace d'un certain espace ultraproduit qui a une structure de L_p-module. On peut alors décrire naturellement l'espace dual de $\mathcal{H}_p^c$ pour $1 < p < \infty$ comme un espace quotient d'un espace ultraproduit, noté $\widetilde{\mathcal{H}}_{p'}^c$, pour $\frac{1}{p}+\frac{1}{p'} = 1$. En établissant qu'en réalité, les espaces $\mathcal{H}_p^c$ et $\widetilde{\mathcal{H}}_p^c$ coïncident on démontre le résultat de dualité suivant.

Théorème 0.3.2. *Soient* $1 < p < \infty$ *et* $\frac{1}{p} + \frac{1}{p'} = 1$. *Alors*

$$(\mathcal{H}_p^c)^* = \mathcal{H}_{p'}^c \quad \textit{avec des normes équivalentes.}$$

Pour $p = 1$, nous étudions le dual de l'espace $\mathcal{H}_1^c$ par une méthode similaire. La définition de l'espace $\mathcal{BMO}^c$ dans ce cas doit être abordée avec prudence. Un candidat naïf pour la norme $\mathcal{BMO}^c$ est donné par

$$\|x\|_{\mathcal{BMO}^c} = \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}, \quad \text{où} \quad \|x\|_{BMO^c(\sigma)} = \sup_{t\in\sigma} \|\mathcal{E}_t(|x - x_{t^-}|^2)\|_\infty^{1/2}.$$

Cependant, dans notre approche, le fait de ne considérer que les partitions finies (alors que dans le cas classique, toutes les partitions aléatoires sont considérées) est restrictif. En effet, si l'une des normes $\|x\|_{BMO^c(\sigma)}$ est finie, alors x est déjà dans $\mathcal{M}$, et il est clair que nous désirons définir un espace $\mathcal{BMO}^c$ plus grand que $\mathcal{M}$. C'est pourquoi on dira qu'un opérateur $x \in L_2(\mathcal{M})$ est dans la boule unité de $\mathcal{BMO}^c$ si on peut l'approcher en norme L_2 par des éléments de la forme

$$w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma \quad \text{dans } L_2(\mathcal{M}) \quad \text{avec} \quad \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{BMO^c(\sigma)} \leq 1.$$

Cette définition, cohérente avec les espaces $\widetilde{\mathcal{H}}_p^c$ évoqués précédemment, permet d'établir l'analogue de la dualité de Fefferman-Stein (0.2.3) dans ce cadre :

$$(\mathcal{H}_1^c)^* = \mathcal{BMO}^c \quad \text{avec des normes équivalentes.}$$

Comme conséquence du Théorème 0.3.2, on obtient que $\mathcal{H}_p^c$ s'injecte dans $L_2(\mathcal{M})$ pour $1 < p < 2$ et dans $L_p(\mathcal{M})$ pour $2 \leq p < \infty$. Cela reste vrai pour $p = 1$, grâce à la propriété de monotonie. On peut ainsi définir l'espace de Hardy $\mathcal{H}_p$ par le même processus que dans le cas discret, en considérant la somme des espaces de Hardy colonne $\mathcal{H}_p^c$ et ligne $\mathcal{H}_p^r$ dans $L_2(\mathcal{M})$ pour $1 \leq p < 2$, et leur intersection dans $L_p(\mathcal{M})$ pour $2 \leq p < \infty$. C'est par une approche d'analyse non standard que l'on obtient finalement l'analogue continu de (0.2.2), dans le sens où l'on démontre dans un premier temps les inégalités de Burkholder-Gundy au niveau des ultraproduits, avant de considérer la limite faible (qui correspond à la partie standard).

Théorème 0.3.3. *Soit* $1 < p < \infty$. *Alors*

$$L_p(\mathcal{M}) = \mathcal{H}_p \quad \textit{avec des normes équivalentes.}$$

Nous examinons également les espaces de Hardy conditionnels h_p. Dans ce cas, on peut encore démontrer une propriété cruciale de monotonie. En considérant le crochet conditionnel

$$\langle x, x\rangle_\sigma = \sum_{t\in\sigma} \mathcal{E}_{t^-}|d_t^\sigma(x)|^2$$

pour une partition finie σ, on définit les espaces de Hardy conditionnels $\widehat{\mathsf{h}}_p^c$ et h_p^c de martingales non commutatives relativement à la filtration $(\mathcal{M}_t)_{0\leq t\leq 1}$. On peut alors adapter la théorie développée pour les espaces $\widehat{\mathcal{H}}_p^c$ et $\mathcal{H}_p^c$ aux espaces $\widehat{\mathsf{h}}_p^c$ et h_p^c pour obtenir de manière similaire

$$\widehat{\mathsf{h}}_p^c = \mathsf{h}_p^c \quad \text{avec des normes équivalentes pour } \ 1 \leq p < \infty.$$

L'analogue conditionnel du Théorème 0.3.2 est également obtenu. Concernant la dualité de Fefferman-Stein, notons que dans ce cas l'espace bmo^c est plus facile à décrire. En effet, il suffit de considérer les opérateurs $x \in L_2(\mathcal{M})$ tels que

$$\sup_{0\leq t\leq 1} \|\mathcal{E}_t|x - \mathcal{E}_t(x)|^2\|_\infty < \infty.$$

Afin d'obtenir l'analogue continu des décompositions (0.2.6) et (0.2.7) pour $1 < p < 2$ et $1 \leq p < 2$ respectivement, comme annoncé précédemment nous introduisons une décomposition de Davis plus forte faisant intervenir un autre espace diagonal $\mathsf{h}_p^{1_c} \subset \mathsf{h}_p^d$. L'intérêt de cet espace $\mathsf{h}_p^{1_c}$ est qu'il présente une certaine propriété de régularité qui permet d'adapter l'approche d'analyse non standard développée dans la preuve du Théorème 0.3.3. Le cas $2 \leq p < \infty$ est ensuite démontré par une approche duale. Malheureusement, il est très difficile de décrire directement l'espace dual de notre analogue continu de l'espace diagonal h_p^d (ou de $\mathsf{h}_p^{1_c}$). C'est pourquoi on introduit une autre variante de la décomposition de Davis dans le cas $1 < p < 2$, qui se base sur une décomposition de Randrianantoanina. Le but de cette approche est de pouvoir remplacer l'espace h_p^d dans la somme par un espace plus grand K_p^d, dont on sait décrire le dual $\mathrm{J}_{p'}^d$. Les analogues continus de (0.2.6) et (0.2.7) sont finalement obtenus en posant

$$\mathsf{h}_p = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c + \mathsf{h}_p^r & \text{pour} \quad 1 \leq p < 2 \\ \mathrm{J}_p^d \cap \mathsf{h}_p^c \cap \mathsf{h}_p^r & \text{pour} \quad 2 \leq p < \infty \end{cases}.$$

Théorème 0.3.4. *Soit* $1 \leq p < \infty$. *Alors*

(i) $\mathcal{H}_p^c = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c & \textit{pour} \quad 1 \leq p < 2 \\ \mathrm{J}_p^d \cap \mathsf{h}_p^c & \textit{pour} \quad 2 \leq p < \infty \end{cases}$ *avec des normes équivalentes.*

(ii) Pour $1 < p < \infty$,

$$L_p(\mathcal{M}) = \mathsf{h}_p \quad \textit{avec des normes équivalentes.}$$

Par approximation, on en déduit une nouvelle caractérisation de l'espace $\mathcal{BMO}^c$.

À la fin du chapitre 3, en se basant sur les approches introduites dans le cas discret évoquée précédemment, nous discutons la décomposition des espaces de Hardy à l'aide d'atomes algébriques, et appliquons cette décomposition pour obtenir l'analogue continu du résultat d'interpolation (0.2.8).

Théorème 0.3.5. *Soit* $1 < p < \infty$. *Alors*

$$\mathcal{H}_p = [\mathcal{BMO}, \mathcal{H}_1]_{\frac{1}{p}} \quad \textit{avec des normes équivalentes.}$$

Cette thèse est constituée de trois chapitres, rédigés en anglais. Le premier chapitre de la thèse présente un article intitulé “A noncommutative Davis’ decomposition for martingales" effectué au début de ma thèse, qui a été publié dans Journal of London Mathematical Society en 2009. Le second chapitre est un travail en collaboration avec Bekjan, Chen et Yin intitulé “Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales", qui a été publié dans Journal of Functional Analysis en 2010. La théorie des espaces de Hardy de martingales non commutatives relativement à une filtration continue fait l’objet du dernier (et plus conséquent) chapitre, qui est le fruit d’une collaboration avec Junge et s’intitule “Theory of $\mathcal{H}_p$-spaces for continuous filtrations in von Neumann algebras".

Introduction

This PhD thesis is part of the theory of noncommutative probability and noncommutative integration. This field is motivated by quantum physic. The main idea of this theory is to replace functions with operators on a Hilbert space and measures by traces. This work deals with noncommutative martingales, and in particular with their associated Hardy spaces. In the probability theory, there are many interactions between martingale inequalities and harmonic analysis. In classical probability, Burkholder, Davis, Gundy, Doob, Meyer, Neveu and many others developed powerful tools like martingale transforms, maximal functions and stopping times, which play an important role in the theory of stochastic processes. However, additional functional analysis and combinatorial tools are needed to extend the classical martingale inequalities to the noncommutative setting. For instance, most of the stopping time arguments are no longer valid in this setting. Moreover, the notion of maximal function cannot be directly extended to operators, since in general we cannot define the supremum of a sequence of operators. The theory of noncommutative martingales has been rapidly developing after Pisier/Xu's seminal paper [35], and nowadays many martingale inequalities have been successfully transferred to the noncommutative setting. The techniques developed in that field may yield new results even in the classical theory, as illustrated in [24]. These investigations also contribute to enrich the knowledge on C^*-algebras and von Neumann algebras, which constitute the setting of the noncommutative theory.

In this introduction, I will first recall some well-known results of the classical theory of discrete time martingales. Then I will consider their noncommutative analogues by recalling the noncommutative martingale inequalities due to Pisier, Xu, Junge, Parcet, Randrianantoanina, Musat and many others. The results obtained with my coauthors Bekjan, Chen, Yin and Junge and presented in this thesis will be detailed in particular. This work essentially concerns the conditioned Hardy spaces of noncommutative martingales. The last part of this introduction deals with the extension of this theory to the noncommutative continuous time martingales, which is a joint work with Marius Junge.

0.1 Classical theory of discrete time martingales

This thesis mainly studies the martingale inequalities and their associated Hardy spaces. In classical probability, the Hardy spaces of martingales are closely related to the Hardy spaces of functions introduced in harmonic analysis. We recall some of their numerous characterizations, which will give the framework of the noncommutative study. We refer to Garsia's book for the theory of martingale inequalities. Let us consider a probability space $(\Omega, \mathcal{F}, \mu)$ equipped with an increasing filtration $(\mathcal{F}_n)_{n\geq 0}$ of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\cup_n \mathcal{F}_n\big)$. Let $(\mathbb{E}_n)_{n\geq 0}$ be the associated sequence of conditional expectations. A martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ is a sequence of random variables

$(f_n)_{n\geq 0}$ in $L_1(\Omega)$ such that

$$\mathbb{E}_n(f_{n+1}) = f_n \quad \text{for all } n \geq 0. \tag{0.1.1}$$

For $1 \leq p \leq \infty$, we say that a martingale f is a bounded $L_p(\Omega)$-martingale if $\|f\|_p = \sup_n \|f_n\|_p < \infty$. For a given bounded $L_1(\Omega)$-martingale $f = (f_n)_{n\geq 0}$, we may consider its square function

$$S(f) = \Big(\sum_n |df_n|^2\Big)^{1/2},$$

where $df_n = f_n - f_{n-1}$, and its maximal function

$$M(f) = \sup_n |f_n|.$$

For $1 \leq p < \infty$, the Hardy space of martingales $H_p(\Omega)$ is defined as the set of bounded $L_p(\Omega)$-martingale f such that $S(f) \in L_p(\Omega)$. We equip this space with the norm

$$\|f\|_{H_p(\Omega)} = \|S(f)\|_p.$$

Let us also introduce the space

$$BMO(\Omega) = \{f \in L_2(\Omega) \ : \ \sup_n \|\mathbb{E}_n|f - f_{n-1}|^2\|_\infty < \infty\}$$

equipped with the norm

$$\|f\|_{BMO(\Omega)} = \sup_n \|\mathbb{E}_n|f - f_{n-1}|^2\|_\infty^{1/2}.$$

This terminology is justified by the fact that for an appropriated choice of $(\Omega, \mathcal{F}, \mu)$ and $(\mathcal{F}_n)_{n\geq 0}$, the space $H_p(\Omega)$ may be identified with the classical Hardy space from function theory and $BMO(\Omega)$ with the class of bounded mean oscillation functions introduced by John and Nirenberg. Fefferman and Stein established the following duality between these two spaces

$$H_1(\Omega)^* = BMO(\Omega). \tag{0.1.2}$$

This duality result will play a fundamental role in the work presented in this thesis. The Hardy space $H_p(\Omega)$ can also be characterized with the maximal function as follows. Let f be a bounded $L_p(\Omega)$-martingale. We say that $f \in H_p^{\max}(\Omega)$ if $M(f) \in L_p(\Omega)$. Using the Burkholder-Davis-Gundy inequalities, which state that for $p \geq 1$ and a bounded $L_p(\Omega)$-martingale f we have

$$\|S(f)\|_p \simeq \|M(f)\|_p, \tag{0.1.3}$$

we deduce that $H_p(\Omega) = H_p^{\max}(\Omega)$ with equivalent norms for $1 \leq p < \infty$. The famous Doob maximal inequality shows that

$$\|M(f)\|_p \leq \delta_p \|f\|_p \quad \text{for } 1 < p \leq \infty, \tag{0.1.4}$$

and by the Burkholder-Gundy inequalities we have

$$\|f\|_p \simeq_{c_p} \|S(f)\|_p \quad \text{for } 1 < p < \infty. \tag{0.1.5}$$

These inequalities mean that for $1 < p < \infty$, the spaces $H_p(\Omega)$ and $H_p^{\max}(\Omega)$ actually coincide with $L_p(\Omega)$.

Martingale transforms are a powerful tool not only in probability but also in several parts of analysis. For instance, Burkholder proved in [3] that martingale transforms are of weak type $(1,1)$, and as application we may deduce other inequalities.

In Burkholder and Gundy's work, some results for the square function $S(f)$ have also been obtained for the conditioned square function

$$s(f) = \Big(\sum_n \mathbb{E}_{n-1}|df_n|^2\Big)^{1/2}.$$

Indeed, the Burkholder inequalities ([4, 5]) establish that

$$\|f\|_p \simeq \Big(\sum_n \|df_n\|_p^p\Big)^{1/p} + \|s(f)\|_p \quad \text{for } 2 \leq p < \infty. \tag{0.1.6}$$

The conditioned square function $s(f)$ plays an important role in Davis' proof of the inequalities (0.1.3) for $p = 1$ ([7]), where the following characterization of $H_1^{\max}(\Omega)$ appears

$$\begin{aligned} M(f) \in L_1(\Omega) \Leftrightarrow \quad & f \text{ decomposes as a sum of two martingales} \\ & f = g + h \text{ satisfying } s(g) \in L_1(\Omega) \text{ and } \sum_n |dh_n| \in L_1(\Omega). \end{aligned} \tag{0.1.7}$$

This decomposition is known as Davis' decomposition. If we denote by $h_1(\Omega)$ the space of L_1-martingales that admit such a decomposition, then it turns out that

$$H_1(\Omega) = h_1(\Omega) = H_1^{\max}(\Omega). \tag{0.1.8}$$

Recall that the dual space of $h_1(\Omega)$ is well-known, and described as the so-called small *bmo* (see [37]).

Other decompositions play an important role in the martingale theory. The Gundy decomposition ([12]) implies for instance the weak type $(1,1)$ boundedness of the square and maximal functions, from which we may deduce some of the inequalities cited previously.

The atomic decomposition is a powerful tool for dealing with duality results, interpolation results and some fundamental inequalities both in martingale theory and harmonic analysis. Atomic decomposition was first introduced in harmonic analysis by Coifman [6]. It is Herz [17] who initiated atomic decomposition for martingale theory. Atoms for martingales are usually defined in terms of stopping times. In order to extend this decomposition to the noncommutative setting, let us mention another approach where the definition of atoms does not involve the notion of stopping times. Indeed, as mentioned previously, the concept of stopping time is, up to now, not well-defined in the generic noncommutative setting. An $\mathcal{F}$-measurable function $a \in L_2(\Omega)$ is said to be an atom if there exist $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

(i) $\mathbb{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Such atoms are called simple atoms by Weisz [50] and are extensively studied by him (see [49] and [50]). In a disguised form in the proof of Theorem A_∞ in [17], Herz establishes an atomic description of the space of predictables martingales $\mathcal{P}_1(\Omega)$. Recall that a martingale $f = (f_n)_{n\geq 0}$ is said to be predictable in L_1 if there exists an adapted sequence $(\lambda_n)_{n\geq -1}$ of non-decreasing, non-negative functions such that $|f_n| \leq \lambda_{n-1}$ for all $n \geq 0$ and $\sup_n \lambda_n \in L_1(\Omega)$. Since $\mathcal{P}_1(\Omega) = \mathcal{H}_1(\Omega)$ for regular martingales, this gives an atomic decomposition of $\mathcal{H}_1(\Omega)$ in the regular case. Such a decomposition is still valid in the general case but for the L_1-martingales f such that the conditioned square function $s(f) \in L_1(\Omega)$ (instead of the square function $S(f)$), as shown by Weisz [49].

0.2 Noncommutative theory of discrete time martingales

We now look at the previous theory in a noncommutative setting, i.e., when we replace functions with operators on a Hilbert space. After describing the construction of the Hardy spaces in this setting and recalling some major results of the theory of noncommutative martingales, I will detail the three points studied in this thesis for a discrete filtration. This work enriches the knowledge of these Hardy spaces, by studying in particular their conditioned versions. It deals with the Davis decomposition, the atomic decomposition and the interpolation of Hardy spaces of noncommutative martingales.

The setting of the theory of noncommutative martingales is given by a von Neumann algebra $\mathcal{M}$, i.e., a unital weak*-closed $*$-algebra of bounded operators on a Hilbert space $\mathcal{H}$. For the sake of simplicity we assume that $\mathcal{M}$ is finite, which means that there exists a normal, faithful and normalized trace τ. Hence $(\mathcal{M}, \tau)$ plays the role of the probability space $(\Omega, \mathcal{F}, \mu)$. The role of the spaces $L_p(\Omega)$ is then played by the noncommutative L_p-spaces $L_p(\mathcal{M}, \tau)$ (see [36]), whose norm is simply given for $1 \leq p < \infty$ by

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p} \quad \text{for } x \in L_p(\mathcal{M}),$$

where $|x| = (x^*x)^{1/2}$ is the usual modulus of x. For $p = \infty$, $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ itself with the operator norm. We also consider an increasing filtration $(\mathcal{M}_n)_{n\geq 0}$ of von Neumann subalgebras of $\mathcal{M}$, and the associated sequence of conditional expectations $(\mathcal{E}_n)_{n\geq 0}$. Armed with this dictionary, we may easily define a noncommutative martingale by simply translating the condition (0.1.1) in this setting. We will say that a sequence $(x_n)_{n\geq 0}$ in $L_1(\mathcal{M})$ is a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$ if

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 0.$$

As said previously, the notions of maximal function and supremum do not take any sense in this setting. Hence, we mainly consider the quadratic Hardy space defined from the square function.

There are many ways of considering the square of an operator x. For instance we may look at the square of the modulus $|x|^2 = x^*x$ and at the square of the modulus of its adjoint $|x^*|^2 = xx^*$. Therefore Pisier and Xu naturally introduced in [35] two square functions, namely the column and row square functions, which define two versions (column and row) of the Hardy space of noncommutative martingales

$$\|x\|_{\mathcal{H}_p^c(\mathcal{M})} = \Big\|\Big(\sum_n |dx_n|^2\Big)^{1/2}\Big\|_p \quad \text{and} \quad \|x\|_{\mathcal{H}_p^r(\mathcal{M})} = \Big\|\Big(\sum_n |dx_n^*|^2\Big)^{1/2}\Big\|_p,$$

where $dx_n = x_n - x_{n-1}$ denotes the martingale difference sequence of the martingale $x = (x_n)_n$. The noncommutative version of the Burkholder-Gundy inequalities proved in [35] is then stated as follows.

$$\|x\|_p \simeq_{c_p} \begin{cases} \max\Big(\Big\|\Big(\sum_n |dx_n|^2\Big)^{1/2}\Big\|_p, \Big\|\Big(\sum_n |dx_n^*|^2\Big)^{1/2}\Big\|_p\Big) & \text{if } 2 \leq p < \infty \\ \inf\Big(\Big\|\Big(\sum_n |dy_n|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n |dz_n^*|^2\Big)^{1/2}\Big\|_p\Big) & \text{if } 1 < p < 2 \end{cases}, \tag{0.2.1}$$

where the infimum runs over all decompositions $dx_n = dy_n + dz_n$ of dx_n as a sum of two martingale difference sequences adapted to the same filtration. This confirms the phenomenon, discovered by Lust-Piquard and Pisier ([29, 30]) when studying the noncommutative Khintchine inequalities, that the martingale inequalities are of different nature according to $p < 2$ or $p > 2$. The Hardy space $\mathcal{H}_p(\mathcal{M})$ is then defined by

$$\mathcal{H}_p(\mathcal{M}) = \begin{cases} \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}) & \text{for } 1 \leq p < 2 \\ \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}) & \text{for } 2 \leq p < \infty \end{cases}.$$

Hence (0.2.1) means that

$$\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M}) \quad \text{with equivalent norms for } 1 < p < \infty. \tag{0.2.2}$$

The weak type $(1,1)$ boundedness of martingale transforms was also established in the noncommutative setting by Randrianantoanina in [39]. In particular, this gives a new proof of (0.2.1) which yields a better constant.

In the Appendix of [35], Pisier and Xu described the dual of the space $\mathcal{H}_1(\mathcal{M})$ as a BMO space, which establishes the noncommutative version of the Fefferman-Stein duality (0.1.2).

$$\mathcal{H}_1(\mathcal{M})^* = \mathcal{BMO}(\mathcal{M}). \tag{0.2.3}$$

Junge and Xu then extended this duality to the case $1 \leq p < 2$ in [24]. Other characterizations of the space $\mathcal{BMO}(\mathcal{M})$ can be found in the noncommutative version of the John-Nirenberg Theorem proved in [22].

Concerning the maximal function, inspired by the noncommutative vector-valued L_p-spaces introduced by Pisier ([34]), Junge translated the notion of the norm of the maximal function to the noncommutative setting and obtained the noncommutative version of the Doob maximal inequality (0.1.4)

$$\|{\sup_n}^+ |\mathcal{E}_n(x)|\|_p \leq \delta_p \|x\|_p \quad \text{for } 1 < p \leq \infty.$$

It is important to note that here $\|{\sup_n}^+|\mathcal{E}_n(x)|\|_p$ is just a notation since $\sup_n |\mathcal{E}_n(x)|$ does not take any sense in the noncommutative setting. With this norm, we may hence define the noncommutative analogue of the maximal Hardy space, denoted by $\mathcal{H}_p^{\max}(\mathcal{M})$. However, it was proved in [25] that the spaces $\mathcal{H}_1(\mathcal{M})$ and $\mathcal{H}_1^{\max}(\mathcal{M})$ do not coincide in general. More precisely $\mathcal{H}_1(\mathcal{M}) \not\subset \mathcal{H}_1^{\max}(\mathcal{M})$. But at the time of this writing we do not know if the reverse inclusion holds in the noncommutative setting.

Junge and Xu extended in [24] the inequalities (0.2.1) to the non tracial case, and proved other noncommutative martingale inequalities. In particular they established the analogue of the Burkholder inequalities (0.1.6) for $2 \leq p < \infty$

$$\|x\|_p \simeq \Big(\sum_n \|dx_n\|_p^p\Big)^{1/p} + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|dx_n|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|dx_n^*|^2\Big)^{1/2}\Big\|_p. \tag{0.2.4}$$

In the same spirit as (0.2.1), these inequalities extend to the case $1 < p < 2$ as follows

$$\|x\|_p \simeq \inf \Big(\Big(\sum_n \|dx_n^d\|_p^p\Big)^{1/p} + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|dx_n^c|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n \mathcal{E}_{n-1}|(dx_n^r)^*|^2\Big)^{1/2}\Big\|_p\Big), \tag{0.2.5}$$

where the infimum is taken over all decompositions $dx_n = dx_n^d + dx_n^c + dx_n^r$ of dx_n as a sum of three martingale difference sequences adapted to the same filtration. Junge and Xu then obtained a new result in the classical theory, and a weak type $(1,1)$ version of this new result in commutative probability was obtained by Parcet in [32]. We may introduce the conditioned column and row square functions, then define the conditioned Hardy spaces

$$\|x\|_{\mathsf{h}_p^c(\mathcal{M})} = \Big\| \Big(\sum_n \mathcal{E}_{n-1} |dx_n|^2 \Big)^{1/2} \Big\|_p \quad , \quad \|x\|_{\mathsf{h}_p^r(\mathcal{M})} = \Big\| \Big(\sum_n \mathcal{E}_{n-1} |dx_n^*|^2 \Big)^{1/2} \Big\|_p$$

and the diagonal Hardy space

$$\|x\|_{\mathsf{h}_p^d(\mathcal{M})} = \Big(\sum_n \|dx_n\|_p^p \Big)^{1/p}.$$

Hence the inequalities (0.2.4) and (0.2.5) can be written as follows

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \quad \text{with equivalent norms for } 1 < p < \infty. \tag{0.2.6}$$

Here the conditioned Hardy space $\mathsf{h}_p(\mathcal{M})$ is defined as the Hardy space $\mathcal{H}_p(\mathcal{M})$, by considering separately the cases $p < 2$ and $p \geq 2$:

$$\mathsf{h}_p(\mathcal{M}) = \begin{cases} \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}) & \text{for } 1 \leq p < 2 \\ \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) & \text{for } 2 \leq p < \infty \end{cases}.$$

Randrianantoanina establishes in [40, 41] the weak type $(1,1)$ versions of the inequalities (0.2.1) and (0.2.4), (0.2.5), by constructing explicit decompositions of martingales. In the classical case, these decompositions are based on stopping times, and Randrianantoanina uses the Cuculescu projections to decompose noncommutative martingales. These projections constitute a useful tool sharing some properties with stopping times, which are sufficient to obtain weak type $(1,1)$ estimates. Using real interpolation techniques, Randrianantoanina deduce some maximal order of the best constants in the inequalities (0.2.1) and (0.2.4), (0.2.5). By similar techniques of decomposition, Parcet and Randrianantoanina constructed a noncommutative Gundy decomposition in [33]. As in the classical case, this decomposition implies new proofs of some noncommutative martingale inequalities.

0.2.1 The Davis decomposition

Combining (0.2.2) with (0.2.6), we obtain that $\mathcal{H}_p(\mathcal{M}) = \mathsf{h}_p(\mathcal{M})$ for $1 < p < \infty$. The main result of chapter 1 is that this equality still holds true for $p = 1$, which means that the first equality of (0.1.8) can be successfully transferred to the noncommutative setting. Note that only weak type $(1,1)$ inequalities was obtained ([40, 41]) in the results recalled previously, and interpolation techniques only imply results for $1 < p < \infty$. Chapter 1 answers positively a question asked in [41]. This can be also considered as a noncommutative analogue of the Davis decomposition with the square function in place of the maximal function. However, the decomposition presented in chapter 1 is not explicit but proved by a dual approach. The dual space of $\mathcal{H}_1(\mathcal{M})$ is well-known, and given by the Fefferman-Stein duality (0.2.3). The missing link was the description of the dual space of $\mathsf{h}_1(\mathcal{M})$ as a BMO space, analogue of the small *bmo* space. We introduce in chapter 1 the following noncommutative *bmo* spaces:
the column bmo space

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ \sup_n \|\mathcal{E}_n |x - x_n|^2\|_\infty < \infty\}$$

equipped with the norm

$$\|x\|_{\mathsf{bmo}^c(\mathcal{M})} = \max\big(\|\mathcal{E}_0(x)\|_\infty \ , \ \sup_n \|\mathcal{E}_n |x - x_n|^2\|_\infty^{1/2}\big),$$

the row bmo space

$$\mathsf{bmo}^r(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ x^* \in \mathsf{bmo}^c(\mathcal{M})\}$$

equipped with the norm

$$\|x\|_{\mathsf{bmo}^r(\mathcal{M})} = \|x^*\|_{\mathsf{bmo}^c(\mathcal{M})},$$

and the diagonal bmo space

$$\mathsf{bmo}^d(\mathcal{M}) = \{\text{sequences of martingale differences in } \ell_\infty(L_\infty(\mathcal{M}))\}$$

equipped with the norm

$$\|x\|_{\mathsf{bmo}^d(\mathcal{M})} = \sup_n \|dx_n\|_\infty.$$

In the spirit of the classical duality presented in [37], we show the following crucial result.

Theorem 0.2.1. *We have* $\mathsf{h}_1(\mathcal{M})^* = \mathsf{bmo}(\mathcal{M})$ *with equivalent norms, where*

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^d(\mathcal{M}) \cap \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}).$$

This conditioned analogue of the Fefferman-Stein duality enlarges the knowledge of the space $\mathsf{h}_1(\mathcal{M})$. This now allows us to use dual approaches in some problems. In particular, observing that the spaces $\mathcal{BMO}(\mathcal{M})$ and $\mathsf{bmo}(\mathcal{M})$ coincide, we then prove the main result of chapter 1.

Theorem 0.2.2. *We have* $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})$ *with equivalent norms.*

These results are then extended to the case $1 < p < 2$ in the same chapter. Using similar arguments, we describe the dual space of $\mathsf{h}_p(\mathcal{M})$ as the space $L_{p'}\mathsf{mo}(\mathcal{M})$ for $\frac{1}{p}+\frac{1}{p'} = 1$, defined similarly to bmo. More precisely, for $2 < q \leq \infty$ we introduce the spaces

$$L_q^c\mathsf{mo}(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ \|{\sup_n}^+ \mathcal{E}_n|x - x_n|^2\|_{q/2} < \infty\}$$

equipped with the norm

$$\|x\|_{L_q^c\mathsf{mo}(\mathcal{M})} = \max\big(\|\mathcal{E}_0(x)\|_q \ , \ \|{\sup_n}^+ \mathcal{E}_n|x - x_n|^2\|_{q/2}^{1/2}\big),$$

$L_q^r\mathsf{mo}(\mathcal{M}) = \{x \in L_2(\mathcal{M}) \ : \ x^* \in L_q^c\mathsf{mo}(\mathcal{M})\}$ and

$$L_q\mathsf{mo}(\mathcal{M}) = \mathsf{h}_q^d(\mathcal{M}) \cap L_q^c\mathsf{mo}(\mathcal{M}) \cap L_q^r\mathsf{mo}(\mathcal{M}).$$

The comparison of the dual spaces then improves the estimate of a constant in the equivalence of the norms $\mathcal{H}_p(\mathcal{M})$ and $\mathsf{h}_p(\mathcal{M})$ for $1 < p < 2$ given in [41]. More precisely, Randrianantoanina obtained $\kappa_p = O((p-1)^{-1})$ as $p \to 1$, where $\|x\|_{\mathsf{h}_p(\mathcal{M})} \leq \kappa_p \|x\|_{\mathcal{H}_p(\mathcal{M})}$, and our approach gives that κ_p remains bounded as $p \to 1$. This dual approach also improves the inequalities (0.2.6), by separating the column and row spaces as follows

$$\mathcal{H}_p^c(\mathcal{M}) = \left\{ \begin{array}{lll} \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) & \text{for} & 1 \leq p < 2 \\ \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) & \text{for} & 2 \leq p < \infty \end{array} \right. . \tag{0.2.7}$$

These inequalities give a more precise description of the relation between the column Hardy spaces. Independently and essentially at the same time, Junge and Mei obtained the same results. They also describe the dual of the spaces $\mathsf{h}_p(\mathcal{M})$ for $1 \leq p < 2$ by using a different method. However, the arguments presented here yield a better constant.

The Davis decomposition plays a fundamental role in the study of the Burkholder inequalities in chapter 3, which deals with the theory of the Hardy spaces of noncommutative martingales with respect to a continuous filtration. Two other approaches of the decomposition (0.2.7) for $1 \leq p < 2$ are also introduced in chapter 3, in order to transfer them to the continuous case. On the one hand, a close look to the dual spaces yields a stronger version of this decomposition. Here the diagonal Hardy space $\mathsf{h}_p^d(\mathcal{M})$ is replaced with a smaller diagonal space called $\mathsf{h}_p^{1_c}(\mathcal{M})$. We then obtain a decomposition which is closer to the classical Davis decomposition (0.1.7). On the other hand, we discuss another variant of this decomposition in the case $1 < p < 2$, based on a deep result of Randrianantoanina ([41]).

0.2.2 The atomic decomposition

The description of the dual space of $\mathsf{h}_1(\mathcal{M})$ established in chapter 1 plays a crucial role in the noncommutative version of the atomic decomposition presented in chapter 2. Indeed, this allows us to adopt a dual approach. We first introduce a notion of noncommutative atom. More precisely, in the spirit of the theory of noncommutative martingales, we define two types of atoms, a column version and a row version. These notions are a direct translation of the definition of simple atom recalled in Section 0.1, by considering respectively the right and left supports of an operator. We say that an operator $a \in L_2(\mathcal{M})$ is a $(1,2)_c$-atom if there exist $n \in \mathbb{N}$ and a projection e in $\mathcal{M}_n$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{-1/2}$.

The main result of the first part of chapter 2 proves that we may decompose the conditioned Hardy space by using these atoms. More precisely, if we denote by $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ (respectively $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$) the atomic space whose unit ball is given by the absolute convex hull of $B_{L_1(\mathcal{M}_0)}$ and $(1,2)_c$-atoms (resp. $(1,2)_r$-atoms), we show the following result.

Theorem 0.2.3. *We have $\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$ with equivalent norms, where*

$$\mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_1^d(\mathcal{M}) + \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}).$$

As the Davis decomposition discussed previously, Theorem 0.2.3 is also proved by a dual approach, similar to that developed in chapter 1 to prove Theorem 0.2.2. The idea is to first describe the dual of the atomic space $\mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$ as a noncommutative Lipschitz space. Then we compare it to the dual of the space $\mathsf{h}_1(\mathcal{M})$, which is now known to be the space $\mathsf{bmo}(\mathcal{M})$. This method does not give an explicit decomposition, but shows that it exists. The Davis decomposition obtained in Theorem 0.2.2 then implies an atomic decomposition of the space $\mathcal{H}_1(\mathcal{M})$.

In chapter 3, we discuss another type of atoms, called algebraic atoms. This decomposition concerns the case $1 \leq p < 2$. We say that an operator $x \in L_2(\mathcal{M})$ is an algebraic $\mathsf{h}_p^c(\mathcal{M})$-atom if we can write

$$x = \sum_n b_n a_n,$$

with

(i) $\mathcal{E}_n(b_n) = 0$ for all $n \geq 0$;

(ii) $a_n \in L_2(\mathcal{M}_n)$ for all $n \geq 0$;

(iii) $\sum_n \|b_n\|_2^2 \leq 1$ and $\Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_q \leq 1$, where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$.

These atoms, less "elementary" than the previous ones, already form an absolutely convex set and are norming for the space $L^c_{p'}\mathsf{mo}$ when p' is the conjugate index of p. By duality, we then show in chapter 3 that the space whose unit ball is given by the absolute convex hull of the algebraic $\mathsf{h}^c_p(\mathcal{M})$-atoms is dense in $\mathsf{h}^c_p(\mathcal{M})$. More generally, the same approach yields a decomposition of larger spaces, called conditioned column L_p-spaces, by using algebraic atoms of the same kind. These new decompositions give additional tools to study the Hardy spaces. They are immediately applied in chapter 3 to obtain interpolation results.

0.2.3 Interpolation of Hardy spaces

Interpolation results for the Hardy spaces $\mathcal{H}_p(\mathcal{M})$ first appear in Musat's paper [31], where she proves for the complex method of interpolation

$$(\mathcal{BMO}(\mathcal{M}), \mathcal{H}_1(\mathcal{M}))_{1/p} = \mathcal{H}_p(\mathcal{M}) \quad \text{for } 1 < p < \infty. \tag{0.2.8}$$

We extend this result to the conditioned Hardy spaces $\mathsf{h}_p(\mathcal{M})$ in chapter 2. Note that our method is much simpler and more elementary than Musat's arguments, and also gives a new proof of (0.2.8). It seems that even in the commutative case, our method is simpler than all existing approaches to the interpolation of Hardy spaces of martingales. The main idea is inspired by an equivalent quasinorm for $h_p(\Omega), 0 < p \leq 2$ introduced by Herz [18] in the commutative case. We translate this quasinorm to the noncommutative setting to obtain a new characterization of $\mathsf{h}_p(\mathcal{M})$, $0 < p \leq 2$, which is more convenient for interpolation.

Theorem 0.2.4. *Let $1 < p < \infty$. Then*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{1/p} = \mathsf{h}_p(\mathcal{M}) \quad \textit{with equivalent norms.}$$

The characterization of $\mathsf{h}_p(\mathcal{M})$ involving the Herz quasinorm gives a new proof of the Fefferman-Stein duality obtained in chapter 1. This approach improves the constants in the equivalence between the norms $\mathsf{h}_p(\mathcal{M})^*$ and $L_{p'}\mathsf{mo}(\mathcal{M})$ for $1 \leq p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For $0 < p < 1$, we discuss the description of the dual space of $\mathsf{h}_p(\mathcal{M})$.

In chapter 3, we develop another approach of the interpolation of Hardy spaces. The idea is to complement Hardy spaces in larger spaces which form an interpolation scale. It is well-known that we may complement the Hardy spaces in the column L_p-spaces for $1 < p < \infty$, by using the Stein projection. However, this does not include the case $p = 1$. We then introduce intermediate spaces, the so-called conditioned column L_p-spaces, in which the Hardy spaces are complemented for $1 \leq p < \infty$. This complementation result is based on a decomposition of the conditioned column L_p-spaces similar to the Davis decomposition discussed in chapter 1, still proved by a dual approach. Using the algebraic atomic decomposition of these spaces, we may show that they form an interpolation scale.

0.3 Noncommutative theory of continuous time martingales

In the third and last chapter, we consider a continuous filtration and study some results cited previously in this setting. In the classical case, the theory of stochastic integrals and martingales with continuous time is a well-known theory with many applications. In the noncommutative setting, there exists a theory of quantum stochastic calculus, which is nowadays only at the algebraic level. In the line of the investigations of the noncommutative martingales detailed previously, chapter 3 studies noncommutative martingales with respect to a continuous filtration in a finite von Neumann algebra. We encounter many difficulties when extending the results from the discrete case to the continuous one. Hence, the theory developed here needs powerful tools of functional analysis like ultraproducts or L_p-modules. The long term goal of this project, in collaboration with Junge, is to develop a satisfactory theory for semimartingales, including the convergence of the stochastic integrals. In the noncommutative setting, we cannot construct the stochastic integrals pathwise as in [8]. It is indeed unimaginable to consider the path of a process of operators in a von Neumann algebra. However, it is well-known that the convergence of the stochastic integrals is closely related to the existence of the quadratic variation bracket $[\cdot,\cdot]$ via the formula

$$f_tg_t = \int^t f_{s^-}dg_s + \int^t g_{s^-}df_s + [f,g]_t.$$

Here the quadratic variation bracket can be characterized as the limit in probability of the following dyadic square functions

$$[f,g]_t = f_0g_0 + \lim_{n\to\infty} \sum_{0\le k<2^n} (f_{t\frac{k+1}{2^n}} - f_{t\frac{k}{2^n}})(g_{t\frac{k+1}{2^n}} - g_{t\frac{k}{2^n}}).$$

Hence we will first study this quadratic variation bracket in the setting of von Neumann algebras, and then deal with stochastic integrals in a forthcoming work based on the theory developed in chapter 3. More precisely, we will focus on the $L_{p/2}$-norm of this bracket by considering the Hardy spaces $H_p(\Omega)$ defined in the classical case by the norm

$$\|f\|_{H_p(\Omega)} = \|[f,f]\|_{p/2}^{1/2}.$$

Chapter 3 study the noncommutative analogue of this space $H_p(\Omega)$ associated to a continuous filtration.

We now consider the same setting as in Section 0.2, i.e., we take a finite von Neumann algebra $(\mathcal{M},\tau)$, equipped in this case with a continuous filtration $(\mathcal{M}_t)_{t\ge 0}$ of subalgebras of $\mathcal{M}$. For simplicity, we assume that the continuous parameter set is given by the interval $[0,1]$. In the spirit of the theory of noncommutative martingales, we should expect to define the bracket $[x,x]$ for a martingale x and then set

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{\widehat{\mathcal{H}}_p^r} = \|[x^*,x^*]\|_{p/2}^{1/2}.$$

Armed with the definition we may attempt to prove the analogue of (0.2.2). We define a candidate for the noncommutative bracket following a nonstandard analysis approach. For a finite partition $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval $[0,1]$ and $x \in \mathcal{M}$, we consider the finite bracket

$$[x,x]_\sigma = \sum_{t\in\sigma} |d_t^\sigma(x)|^2,$$

where $d_t^\sigma(x) = \mathcal{E}_t(x) - \mathcal{E}_{t^-}(x)$ and for $t = t_j$ $(j = 1,\cdots,n)$, $t^- = t_{j-1}$ denotes its predecessor in the partition σ. By convention we set $d_0^\sigma(x) = \mathcal{E}_0(x)$. Then for $p > 2$, (0.2.2) gives an a-priori bound $\|[x,x]_\sigma]\|_{p/2}^{1/2} \le c_p\|x\|_p$. Hence, for a fixed ultrafilter $\mathcal{U}$ refining the general net of finite partitions of $[0,1]$, we may simply define

$$[x,x]_{\mathcal{U}} = w\text{-}\lim_{\sigma,\mathcal{U}} [x,x]_\sigma,$$

where the weak-limit is taken in the reflexive space $L_{p/2}(\mathcal{M})$. In fact, in nonstandard analysis, the weak-limit corresponds to the standard part and is known to coincide with the classical definition of the bracket for commutative martingales. However, the norm is only lower semi-continuous with respect to the weak topology and we should not expect Burkholder/Gundy inequalities for continuous filtrations to be a simple consequence of the discrete theory of the Hardy spaces. Yet, using the crucial observation that the $L_{p/2}$-norms of the discrete brackets $[x,x]_\sigma$ are monotonous up to a constant, we may show the following result.

Theorem 0.3.1. *Let $1 \leq p < \infty$ and $x \in \mathcal{M}$. Then*

$$\|[x,x]_{\mathcal{U}}\|_{p/2} \simeq \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2} \simeq \begin{cases} \sup_\sigma \|[x,x]_\sigma\|_{p/2} & \text{for} \quad 1 \leq p < 2 \\ \inf_\sigma \|[x,x]_\sigma\|_{p/2} & \text{for} \quad 2 \leq p < \infty \end{cases}.$$

In particular, this implies that the $L_{p/2}$-norm of the bracket $[x,x]_{\mathcal{U}}$ does not depend on the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm. The independence of the bracket $[x,x]_{\mathcal{U}}$ itself from the choice of $\mathcal{U}$ will be discussed in a forthcoming work. Hence for $1 \leq p < \infty$ and $x \in \mathcal{M}$ we define the norms

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]_{\mathcal{U}}\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2}^{1/2} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

We denote by $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ respectively the corresponding completions. These two processes actually define the same space:

$$\widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c \quad \text{with equivalent norms for} \;\; 1 \leq p < \infty.$$

The proof is based on Theorem 0.3.1 and follows two different approaches according to the value of p. For $2 \leq p < \infty$, we use complementation arguments. The case $1 \leq p < 2$ is obtained by duality. We embed $\widehat{\mathcal{H}}_p^c$ in a larger ultraproduct space in order to study its dual space. Hence this defines a good candidate for the Hardy space of noncommutative martingales with respect to the continuous filtration $(\mathcal{M}_t)_{0 \leq t \leq 1}$. We now want to establish for this space the analogues of many results cited in Section 0.2. For doing this, we will use the definition of the space $\mathcal{H}_p^c$, which will be more practical to work with. In particular, we may consider $\mathcal{H}_p^c$ as a subspace of some ultraproduct space, which has an L_p-module structure. Hence we may describe in a natural way the dual space of $\mathcal{H}_p^c$ for $1 < p < \infty$ as a quotient space of an ultraproduct space, denoted by $\widetilde{\mathcal{H}}_{p'}^c$, for $\frac{1}{p} + \frac{1}{p'} = 1$. Proving that the spaces $\mathcal{H}_p^c$ and $\widetilde{\mathcal{H}}_p^c$ actually coincide, we show the following duality result.

Theorem 0.3.2. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$(\mathcal{H}_p^c)^* = \mathcal{H}_{p'}^c \quad \textit{with equivalent norms.}$$

For $p = 1$, we study the dual space of $\mathcal{H}_1^c$ by a similar method. We have to be careful when defining the space $\mathcal{BMO}^c$. A naive candidate for the $\mathcal{BMO}^c$ norm is given by

$$\|x\|_{\mathcal{BMO}^c} = \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}, \quad \text{where} \quad \|x\|_{BMO^c(\sigma)} = \sup_{t \in \sigma} \|\mathcal{E}_t(|x - x_{t^-}|^2)\|_\infty^{1/2}.$$

However, here our restriction to finite partitions (instead of random partitions in the classical case) is restrictive. Indeed, if one of the $\|x\|_{BMO^c(\sigma)}$'s is finite, then x is already in $\mathcal{M}$. Definitively, we expect $\mathcal{BMO}^c$ to be larger than $\mathcal{M}$. We will therefore say that an element $x \in L_2(\mathcal{M})$ belongs to the unit ball of $\mathcal{BMO}^c$ if it can be approximated in L_2-norm by elements of the form

$$w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma \quad \text{in } L_2(\mathcal{M}) \quad \text{with} \quad \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{BMO^c(\sigma)} \leq 1.$$

This definition, consistent with the spaces $\widetilde{\mathcal{H}}_p^c$ introduced previously, yields the analogue of the Fefferman-Stein duality (0.2.3) in this setting:

$$(\mathcal{H}_1^c)^* = \mathcal{BMO}^c \quad \text{with equivalent norms.}$$

As a consequence of Theorem 0.3.2, $\mathcal{H}_p^c$ embeds into $L_2(\mathcal{M})$ for $1 < p < 2$ and into $L_p(\mathcal{M})$ for $2 \leq p < \infty$. In fact, this still holds true for $p = 1$ by the monotonicity property. Hence we may define the Hardy space $\mathcal{H}_p$ as in the discrete setting by considering the sum of the column and row Hardy spaces in $L_2(\mathcal{M})$ for $1 \leq p < 2$, and their intersection in $L_p(\mathcal{M})$ for $2 \leq p < \infty$. The continuous analogue of (0.2.2) is then obtained by a nonstandard analysis approach, i.e., we first prove the Burkolder-Gundy inequalities at the ultraproduct level, and then take the weak limit (i.e., the standard part).

Theorem 0.3.3. *Let $1 < p < \infty$. Then*

$$L_p(\mathcal{M}) = \mathcal{H}_p \quad \textit{with equivalent norms.}$$

We are also interested in the conditioned Hardy spaces h_p. In this case, we still have a crucial monotonicity property, and considering the conditioned bracket

$$\langle x, x \rangle_\sigma = \sum_{t \in \sigma} \mathcal{E}_{t^-} |d_t^\sigma(x)|^2$$

for a finite partition σ, we define the conditioned Hardy spaces $\widehat{\mathsf{h}}_p^c$ and h_p^c of noncommutative martingales with respect to the filtration $(\mathcal{M}_t)_{0 \leq t \leq 1}$. Then we may adapt the theory developed for the spaces $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ to the spaces $\widehat{\mathsf{h}}_p^c$ and h_p^c and obtain similarly that

$$\widehat{\mathsf{h}}_p^c = \mathsf{h}_p^c \quad \text{with equivalent norms for } \ 1 \leq p < \infty.$$

Moreover, we can prove the conditioned analogue of Theorem 0.3.2. Concerning the Fefferman-Stein duality, note that in this case the space bmo^c is easier to describe. It is defined as the set of operators $x \in L_2(\mathcal{M})$ such that

$$\sup_{0 \leq t \leq 1} \|\mathcal{E}_t |x - x_t|^2\|_\infty < \infty.$$

To obtain the continuous analogue of the decompositions (0.2.6) and (0.2.7) for $1 < p < 2$ and $1 \leq p < 2$ respectively, as announced previously we need to introduce a stronger Davis decomposition involving another diagonal space $h_p^{1_c} \subset h_p^d$. This space $h_p^{1_c}$ presents the advantage that it satisfies a certain regularity property, which is useful to adapt the nonstandard analysis approach developed in the proof of Theorem 0.3.3. The case $2 \leq p < \infty$ is then obtained by a dual approach. Unfortunately, we cannot directly describe the dual spaces of our continuous analogues of the diagonal spaces h_p^d and $\mathsf{h}_p^{1_c}$. This is why we introduce a variant of the Davis decomposition for $1 < p < 2$, based on a deep result of Randrianantoanina. This new decomposition will allow us to replace h_p^d in the sum with a larger space K_p^d. We may now describe the dual space of K_p^d, and denote it by $\mathrm{J}_{p'}^d$. The continuous analogues of (0.2.6) and (0.2.7) finally follow by setting

$$\mathsf{h}_p = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c + \mathsf{h}_p^r & \text{for} \quad 1 \leq p < 2 \\ \mathrm{J}_p^d \cap \mathsf{h}_p^c \cap \mathsf{h}_p^r & \text{for} \quad 2 \leq p < \infty \end{cases}.$$

Theorem 0.3.4. *Let* $1 \leq p < \infty$. *Then*

(i) $\mathcal{H}_p^c = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c & \text{for } 1 \leq p < 2 \\ \mathsf{J}_p^d \cap \mathsf{h}_p^c & \text{for } 2 \leq p < \infty \end{cases}$ *with equivalent norms.*

(ii) For $1 < p < \infty$,

$$L_p(\mathcal{M}) = \mathsf{h}_p \quad \textit{with equivalent norms.}$$

By approximation, we deduce a new characterization of $\mathcal{BMO}^c$.

At the end of chapter 3, based on the approaches introduced in Section 0.2 in the discrete case, we discuss the decomposition of the Hardy spaces into algebraic atoms, and we use this decomposition to obtain the continuous analogue of the interpolation result (0.2.8).

Theorem 0.3.5. *Let* $1 < p < \infty$. *Then*

$$\mathcal{H}_p = [\mathcal{BMO}, \mathcal{H}_1]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

This thesis is decomposed into three chapters, written in english. The first chapter presents a paper entitled "A noncommutative Davis' decomposition for martingales" done at the beginning of my thesis. It was published in Journal of London Mathematical Society in 2009. The second chapter is a joint work with Bekjan, Chen and Yin entitled "Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales". It was published in Journal of Functional Analysis in 2010. The theory of the Hardy spaces of noncommutative martingales with respect to a continuous filtration is the subject of the last (and more substantial) chapter, which is a collaboration with Junge and is entitled "Theory of $\mathcal{H}_p$-spaces for continuous filtrations in von Neumann algebras".

Chapter 1

A noncommutative Davis' decomposition for martingales

Introduction

The theory of noncommutative martingale inequalities has been rapidly developed since the establishment of the noncommutative Burkholder-Gundy inequalities in [35]. Many of the classical martingale inequalities has been transferred to the noncommutative setting. These include, in particular, the Doob maximal inequality in [20], the Burkholder/Rosenthal inequality in [24], [27], several weak type $(1,1)$ inequalities in [39, 40, 41] and the Gundy decomposition in [33]. We would point out that the noncommutative Gundy's decomposition in this last work is remarkable and powerful in the sense that it implies several previous inequalities. For instance, it yields quite easily Randrianantoanina's weak type $(1,1)$ inequality on martingale transforms (see [33]). It is, however, an open problem weather there exist a noncommutative analogue of the classical Davis' decomposition for martingales (see [41] and [32]). This is the main concern of our paper.

We now recall the classical Davis' decomposition for commutative martingales. Given a probability space (Ω, A, μ), let $A_1, A_2, \cdots$ be an increasing filtration of σ-subalgebras of A and let $\mathbb{E}_1, \mathbb{E}_2, \cdots$ denote the corresponding family of conditional expectations. Let $f = (f_n)_{n\geq 1}$ be a martingale adapted to this filtration and bounded in $L_1(\Omega)$. Then $M(f) = \sup |f_n| \in L_1(\Omega)$ iff we can decompose f as a sum $f = g + h$ of two martingales adapted to the same filtration and satisfaying

$$s(g) = \Big(\sum_{n=1}^{\infty} \mathbb{E}_{n-1} |dg_n|^2 \Big)^{1/2} \in L_1(\Omega) \quad \text{and} \quad \sum_{n=1}^{\infty} |dh_n| \in L_1(\Omega).$$

We refer to [10] and [7] for more information.

We denote by h_1 the space of martingales f with respect to $(A_n)_{n\geq 1}$ which admit such a decomposition and by $H_1^{\max}$ the space of martingales such that $M(f) \in L_1(\Omega)$. This decomposition appeared for the first time in [7] where Davis applied it to prove his famous theorem on the equivalence in L_1-norm between the martingale square function and Doob's maximal function:

$$\|M(f)\|_1 \approx \|S(f)\|_1$$

where $S(f) = \Big(\sum_{n\geq 1} |df_n|^2 \Big)^{1/2}$. If we denote by H_1 the space of all L_1-martingales f such that

$S(f) \in L_1(\Omega)$, then it turns out that the Hardy space H_1 coincides with the other two Hardy spaces:

$$H_1 = \mathsf{h}_1 = H_1^{\max}.$$

The main result of this paper is that the equality $H_1 = \mathsf{h}_1$ holds in the noncommutative case. This answers positively a question asked in [41]. This can be also considered as a noncommutative analogue of Davis' decomposition with the square function in place of the maximal function. Our approach to this result is via duality. We describe the dual space of h_1 as a $\mathcal{BMO}$ type space. This is the second main result of the paper. Recall that this latter result is well known in the commutative case, the resulting dual of h_1 is then the so-called small bmo (see [37]). Combining this duality with that between $\mathcal{H}_1$ and $\mathcal{BMO}$ established in [35], we otain the announced equality $\mathcal{H}_1 = \mathsf{h}_1$ in the noncommutative setting.

Concerning $\mathcal{H}_1^{\max}$, it is shown in [25], Corollary 14, that $\mathcal{H}_1$ and $\mathcal{H}_1^{\max}$ do not coincide in general. More precisely $\mathcal{H}_1 \not\subset \mathcal{H}_1^{\max}$. But at the time of this writing we do not know if the reverse inclusion holds in the noncommutative setting.

The paper is organized as follows: in Section 1 we give some preliminaries on noncommutative martingales and the noncommutative Hardy spaces. Section 2 is devoted to the determination of the dual of h_1, which allows us to show the equality $\mathcal{H}_1 = \mathsf{h}_1$. This duality is extended to the case $1 < p < 2$ in Section 3. There we describe the dual of h_p and use it to improve the estimation of an equivalence constant in the equivalence of the norms h_p and $\mathcal{H}_p$ given in [41].

After completing this paper, we learnt that Junge and Mei obtained the main result essentially at the same time (see Lemma 1.1 of [21]). Note, however, that our proof of one direction in the duality theorem is different from theirs and yields a better constant (see Remark 1.3.2).

1.1 Preliminaries

We use standard notation in operator algebras. We refer to [28] and [46] for background on von Neumann algebra theory. Throughout the paper all von Neumann algebras are assumed to be finite. Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful normalized trace τ. For $1 \leq p \leq \infty$, we denote by $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. Note that if $p = \infty$, $L_p(\mathcal{M})$ is just $\mathcal{M}$ itself with the operator norm; also recall that the norm in $L_p(\mathcal{M})$ $(1 \leq p < \infty)$ is defined as

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L_p(\mathcal{M})$$

where

$$|x| = (x^*x)^{1/2}$$

is the usual modulus of x. We refer to the survey [36] for more information on noncommutative L_p-spaces.

We now turn to the definition of noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$. $(\mathcal{M}_n)_{n\geq 1}$ is called a filtration of $\mathcal{M}$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. Let $\mathcal{E}_n = \mathcal{E}(\,\cdot\,|\mathcal{M}_n)$ be the trace preserving conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. $\mathcal{E}_n$ defines a norm 1 projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$ for all

$1 \le p \le \infty$, and $\mathcal{E}_n(x) \ge 0$ whenever $x \ge 0$. A noncommutative martingale with respect to $(\mathcal{M}_n)_{n\ge1}$ is a sequence $x = (x_n)_{n\ge1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n, \quad \forall n \ge 1.$$

If additionally, $x \in L_p(\mathcal{M})$ for some $1 \le p \le \infty$, then x is called an L_p-martingale. In this case, we set

$$\|x\|_p = \sup_{n\ge1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called a bounded L_p-martingale. The difference sequence $dx = (dx_n)_{n\ge1}$ of a martingale $x = (x_n)_{n\ge1}$ is defined by

$$dx_n = x_n - x_{n-1}$$

with the usual convention that $x_0 = 0$.

We now describe Hardy spaces of noncommutative martingales. Following [35], for $1 \le p < \infty$ and any finite sequence $a = (a_n)_{n\ge1}$ in $L_p(\mathcal{M})$, we set

$$\|a\|_{L_p(\mathcal{M};\ell_2^c)} = \Big\|\Big(\sum_{n\ge1} |a_n|^2\Big)^{1/2}\Big\|_p, \quad \|a\|_{L_p(\mathcal{M};\ell_2^r)} = \Big\|\Big(\sum_{n\ge1} |a_n^*|^2\Big)^{1/2}\Big\|_p.$$

Then $\|\cdot\|_{L_p(\mathcal{M};\ell_2^c)}$ (resp. $\|\cdot\|_{L_p(\mathcal{M};\ell_2^r)}$) defines a norm on the family of finite sequences of $L_p(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L_p(\mathcal{M};\ell_2^c)$ (resp. $L_p(\mathcal{M};\ell_2^r)$). For $p = \infty$, we define $L_\infty(\mathcal{M};\ell_2^c)$ (respectively $L_\infty(\mathcal{M};\ell_2^r)$) as the Banach space of the sequences in $L_\infty(\mathcal{M})$ such that $\sum_{n\ge1} x_n^* x_n$ (respectively $\sum_{n\ge1} x_n x_n^*$) converge for the weak operator topology. We recall the two square functions introduced in [35]. Let $x = (x_n)_{n\ge1}$ be an L_p-martingale. We define

$$S_{c,n}(x) = \Big(\sum_{k=1}^n |dx_k|^2\Big)^{1/2} \quad \text{and} \quad S_{r,n}(x) = \Big(\sum_{k=1}^n |dx_k^*|^2\Big)^{1/2}.$$

If $dx \in L_p(\mathcal{M};\ell_2^c)$ (resp. $dx \in L_p(\mathcal{M};\ell_2^r)$), we set

$$S_c(x) = \Big(\sum_{k\ge1} |dx_k|^2\Big)^{1/2} \quad \Big(\text{resp.} \quad S_r(x) = \Big(\sum_{k\ge1} |dx_k^*|^2\Big)^{1/2}\Big).$$

Then $S_c(x)$ and $S_r(x)$ are elements in $L_p(\mathcal{M})$. Note that $dx \in L_p(\mathcal{M};\ell_2^c)$ if and only if the sequence $(S_{c,n}(x))_{n\ge1}$ is bounded in $L_p(\mathcal{M})$. In this case

$$S_c(x) = \lim_{n\to\infty} S_{c,n}(x) \quad \text{(relative to the weak}^*\text{-topology for } p = \infty).$$

The same remark applies to the row square function.

Let $1 \le p < \infty$. Define $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) to be the space of all L_p-martingales with respect to $(\mathcal{M}_n)_{n\ge1}$ such that $dx \in L_p(\mathcal{M};\ell_2^c)$ (resp. $dx \in L_p(\mathcal{M};\ell_2^r)$), and set

$$\|x\|_{\mathcal{H}_p^c(\mathcal{M})} = \|dx\|_{L_p(\mathcal{M};\ell_2^c)} \quad \text{and} \quad \|x\|_{\mathcal{H}_p^r(\mathcal{M})} = \|dx\|_{L_p(\mathcal{M};\ell_2^r)}.$$

Equipped respectively with the previous norms, $\mathcal{H}_p^c(\mathcal{M})$ and $\mathcal{H}_p^r(\mathcal{M})$ are Banach spaces.

Then we define the *Hardy space of noncommutative martingales* as follows:
if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the sum norm

$$\|x\|_{\mathcal{H}_p(\mathcal{M})} = \inf\{\|y\|_{\mathcal{H}_p^c(\mathcal{M})} + \|z\|_{\mathcal{H}_p^r(\mathcal{M})} : x = y + z, y \in \mathcal{H}_p^c(\mathcal{M}), z \in \mathcal{H}_p^r(\mathcal{M})\};$$

if $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the intersection norm

$$\|x\|_{\mathcal{H}_p(\mathcal{M})} = \max\left(\|x\|_{\mathcal{H}_p^c(\mathcal{M})}\ ,\ \|x\|_{\mathcal{H}_p^r(\mathcal{M})}\right).$$

We now consider the conditioned versions of square functions and Hardy spaces developed in [24]. Let $1 \leq p < \infty$. For a finite L_∞-martingale $x = (x_n)_{n\geq 1}$, define (with $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|x\|_{\mathsf{h}_p^c(\mathcal{M})} = \Big\|\Big(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|dx_n|^2)\Big)^{1/2}\Big\|_p$$

and

$$\|x\|_{\mathsf{h}_p^r(\mathcal{M})} = \Big\|\Big(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|dx_n^*|^2)\Big)^{1/2}\Big\|_p.$$

Let $\mathsf{h}_p^c(\mathcal{M})$ and $\mathsf{h}_p^r(\mathcal{M})$ be the corresponding completions. Then $\mathsf{h}_p^c(\mathcal{M})$ and $\mathsf{h}_p^r(\mathcal{M})$ are Banach spaces. We define the column and row conditioned square functions as follows. For any finite martingale $x = (x_n)_{n\geq 1}$ in $L_2(\mathcal{M})$, set

$$s_c(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}(|dx_n|^2)\Big)^{1/2} \quad \text{and} \quad s_r(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}(|dx_n^*|^2)\Big)^{1/2}.$$

Then

$$\|x\|_{\mathsf{h}_p^c(\mathcal{M})} = \|s_c(x)\|_p \quad \text{and} \quad \|x\|_{\mathsf{h}_p^r(\mathcal{M})} = \|s_r(x)\|_p.$$

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty.$$

Set

$$s_d(x) = \Big(\sum_{n\geq 1} |dx_n|_p^p\Big)^{1/p}.$$

We note that

$$\|s_d(x)\|_p = \|dx\|_{\ell_p(L_p(\mathcal{M}))}.$$

Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.

Following [24], we define the conditioned version of martingale Hardy spaces as follows:
if $1 \leq p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p(\mathcal{M})} = \inf\{\|x^d\|_{\mathsf{h}_p^d(\mathcal{M})} + \|x^c\|_{\mathsf{h}_p^c(\mathcal{M})} + \|x^r\|_{\mathsf{h}_p^r(\mathcal{M})}\}$$

where the infimum is taken over all decompositions $x = x^d + x^c + x^r$ with $x^k \in \mathsf{h}_p^k(\mathcal{M}), k \in \{d, c, r\}$;
if $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{h_p(\mathcal{M})} = \max\left(\|x\|_{\mathsf{h}_p^d(\mathcal{M})}, \|x\|_{\mathsf{h}_p^c(\mathcal{M})}, \|x\|_{\mathsf{h}_p^r(\mathcal{M})}\right).$$

Throughout the rest of the paper letters like $\kappa_p, \nu_p \cdots$ will denote positive constants, which depend only on p and may change from line to line. We will write $a_p \approx b_p$ as $p \to p_0$ to abbreviate the statement that there are two absolute positive constants K_1 and K_2 such that

$$K_1 \leq \frac{a_p}{b_p} \leq K_2 \quad \text{for } p \text{ close to } p_0.$$

1.2 Noncommutative Davis' decomposition and the dual of h_1

Now we can state the main result of this section announced previously in introduction.

Theorem 1.2.1. *We have*

$$\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M}) \quad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathcal{H}_1(\mathcal{M})$,

$$\frac{1}{2}\|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathcal{H}_1} \leq \sqrt{6}\|x\|_{\mathsf{h}_1}.$$

The inclusion $h_1(\mathcal{M}) \subset \mathcal{H}_1(\mathcal{M})$ directly comes from the dual form of the reverse noncommutative Doob inequality in the case $0 < p < 1$ proved in [24], which is stated as follows. For all finite sequences $a = (a_n)_{n\geq 1}$ of positive elements in $L_p(\mathcal{M})$,

$$\Big\|\sum_{n\geq 1} a_n\Big\|_p \leq 2^{1/p}\Big\|\sum_{n\geq 1}\mathcal{E}_{n-1}(a_n)\Big\|_p.$$

Indeed, applying to $p = 1/2$ and $a_n = |dx_n|^2$, we obtain for any martingale x in L_p

$$\begin{aligned}
\Big\|\Big(\sum_{n\geq 1}\mathcal{E}_{n-1}(|dx_n|^2)\Big)^{1/2}\Big\|_1 &= \Big\|\sum_{n\geq 1}\mathcal{E}_{n-1}(|dx_n|^2)\Big\|_{1/2}^{1/2} \\
&\geq \frac{1}{4}\Big\|\sum_{n\geq 1}|dx_n|^2\Big\|_{1/2}^{1/2} \\
&= \frac{1}{4}\Big\|\Big(\sum_{n\geq 1}|dx_n|^2\Big)^{1/2}\Big\|_1;
\end{aligned}$$

so $\|S_c(x)\|_1 \leq 4\|s_c(x)\|_1$. Similarly $\|S_r(x)\|_1 \leq 4\|s_r(x)\|_1$. On the other hand, we have

$$\begin{aligned}
\|S_c(x)\|_1 &= \Big\|\sum_{n\geq 1}|dx_n|^2\Big\|_{1/2}^{1/2} \\
&\leq \sum_{n\geq 1}\||dx_n|^2\|_{1/2}^{1/2} \\
&= \sum_{n\geq 1}\|dx_n\|_1 = \|s_d(x)\|_1.
\end{aligned}$$

Thus if $x \in h_1(\mathcal{M})$, there exists $(x^d, x^c, x^r) \in h_1^d(\mathcal{M}) \times h_1^c(\mathcal{M}) \times h_1^r(\mathcal{M})$ such that $x = x^d + x^c + x^r$ and from above $x^c, x^d \in \mathcal{H}_1^c(\mathcal{M})$ and $x^r \in \mathcal{H}_1^r(\mathcal{M})$, so $x \in \mathcal{H}_1(\mathcal{M})$. Hence we deduce

$$\|x\|_{\mathcal{H}_1} \leq 4\|x\|_{\mathsf{h}_1}.$$

For the reverse inclusion, we will show the dual version. The dual approach gives also another proof for the direct inclusion, with a constant $\sqrt{6}$ instead of 4. Recall that the dual space of $\mathcal{H}_1(\mathcal{M})$ is the space $\mathcal{BMO}(\mathcal{M})$ defined as follows (we refer to [35] for details). Set

$$\mathcal{BMO}^c(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : \sup_{n\geq 1} \|\mathcal{E}_n(|a - \mathcal{E}_{n-1}(a)|^2)\|_\infty < \infty\}$$

where, as usual, $\mathcal{E}_0(a) = 0$. $\mathcal{BMO}^c(\mathcal{M})$ is equipped with the norm

$$\|a\|_{\mathcal{BMO}^c(\mathcal{M})} = \Big(\sup_{n\geq 1} \|\mathcal{E}_n(|a - \mathcal{E}_{n-1}(a)|^2)\|_\infty\Big)^{1/2}.$$

Then $(\mathcal{BMO}^c(\mathcal{M}), \|\cdot\|_{\mathcal{BMO}^c(\mathcal{M})})$ is a Banach space. Similarly, we define

$$\mathcal{BMO}^r(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : a^* \in \mathcal{BMO}^c(\mathcal{M})\}$$

equipped with the norm

$$\|a\|_{\mathcal{BMO}^r(\mathcal{M})} = \|a^*\|_{\mathcal{BMO}^c(\mathcal{M})}.$$

Finally, we set

$$\mathcal{BMO}(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M}) \cap \mathcal{BMO}^r(\mathcal{M})$$

equipped with the intersection norm

$$\|a\|_{\mathcal{BMO}(\mathcal{M})} = \max\big(\|a\|_{\mathcal{BMO}^c(\mathcal{M})}, \|a\|_{\mathcal{BMO}^r(\mathcal{M})}\big).$$

Note that if $a_n = \mathcal{E}_n(a)$, then

$$\mathcal{E}_n(|a - \mathcal{E}_{n-1}(a)|^2) = \mathcal{E}_n\Big(\sum_{k\geq n} |da_k|^2\Big).$$

To describe the dual space of $\mathsf{h}_1(\mathcal{M})$, we introduce similar spaces $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{bmo}^r(\mathcal{M})$. Let

$$\mathsf{bmo}^c(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : \sup_{n\geq 1} \|\mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_\infty < \infty\}$$

and equip $\mathsf{bmo}^c(\mathcal{M})$ with the norm

$$\|a\|_{\mathsf{bmo}^c(\mathcal{M})} = \max\Big(\|\mathcal{E}_1(a)\|_\infty \;,\; \Big(\sup_{n\geq 1} \|\mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_\infty\Big)^{1/2}\Big).$$

This is a Banach space. Similarly, we define

$$\mathsf{bmo}^r(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : a^* \in \mathsf{bmo}^c(\mathcal{M})\}$$

equipped with the norm

$$\|a\|_{\mathsf{bmo}^r(\mathcal{M})} = \|a^*\|_{\mathsf{bmo}^c(\mathcal{M})}.$$

For any sequence $a = (a_n)_{n\geq 1}$ in $\mathcal{M}$, we set

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_{n\geq 1} \|a_n\|_\infty.$$

Let $\mathsf{bmo}^d(\mathcal{M})$ be the subspace of $\ell_\infty(L_\infty(\mathcal{M}))$ consisting of all martingale difference sequences.
Finally, we set

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

equipped with the intersection norm

$$\|a\|_{\mathsf{bmo}(\mathcal{M})} = \max\left(\|a\|_{\mathsf{bmo}^c(\mathcal{M})}, \|a\|_{\mathsf{bmo}^r(\mathcal{M})}, \|a\|_{\mathsf{bmo}^d(\mathcal{M})}\right).$$

Note that $\mathsf{bmo}^c(\mathcal{M}), \mathsf{bmo}^r(\mathcal{M})$ and $\mathsf{bmo}(\mathcal{M}) \subset L_2(\mathcal{M})$. As before, we have

$$\mathcal{E}_n(|a - \mathcal{E}_n(a)|^2) = \mathcal{E}_n\Big(\sum_{k>n} |da_k|^2\Big).$$

For convenience we denote $\mathcal{H}_1^c(\mathcal{M}), \mathcal{BMO}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}), \mathsf{bmo}^c(\mathcal{M}) \cdots$, respectively, by $\mathcal{H}_1^c, \mathcal{BMO}^c, \mathsf{h}_1^c, \mathsf{bmo}$
The relation between the spaces $\mathcal{BMO}$ and bmo can be stated as follows.

Proposition 1.2.2. *We have*

$$\begin{array}{rcl} \mathcal{BMO}^c & = & \mathsf{bmo}^c \cap \mathsf{bmo}^d, \\ \mathcal{BMO}^r & = & \mathsf{bmo}^r \cap \mathsf{bmo}^d, \\ \mathcal{BMO} & = & \mathsf{bmo}. \end{array}$$

More precisely, for any $a \in L_2(\mathcal{M})$,

$$\|a\|_{\mathsf{bmo}^c \cap \mathsf{bmo}^d} \leq \|a\|_{\mathcal{BMO}^c} \leq \sqrt{2}\|a\|_{\mathsf{bmo}^c \cap \mathsf{bmo}^d}$$

and similar inequalities hold for the two other spaces.

Proof. Let $a \in \mathcal{BMO}^c$. Then

$$\Big\|\mathcal{E}_n\Big(\sum_{k>n} |da_k|^2\Big)\Big\|_\infty \leq \Big\|\mathcal{E}_n\Big(\sum_{k\geq n} |da_k|^2\Big)\Big\|_\infty$$

and

$$\|da_n\|_\infty^2 = \|\mathcal{E}_n |da_n|^2\|_\infty \leq \Big\|\mathcal{E}_n\Big(\sum_{k\geq n} |da_k|^2\Big)\Big\|_\infty.$$

Since $da_1 = \mathcal{E}_1(a)$, taking the supremum over all $n \geq 1$ we find

$$\|a\|_{\mathsf{bmo}^c \cap \mathsf{bmo}^d} \leq \|a\|_{\mathcal{BMO}^c}.$$

Conversely, let $a \in \mathsf{bmo}^c \cap \mathsf{bmo}^d$, then

$$\Big\|\mathcal{E}_n\Big(\sum_{k\geq n} |da_k|^2\Big)\Big\|_\infty \leq \Big\|\mathcal{E}_n\Big(\sum_{k>n} |da_k|^2\Big)\Big\|_\infty + \|da_n\|_\infty^2.$$

Taking the supremum over all $n \geq 1$ we obtain

$$\|a\|_{\mathcal{BMO}^c}^2 \leq \|a\|_{\mathsf{bmo}^c}^2 + \|a\|_{\mathsf{bmo}^d}^2 \leq 2\|a\|_{\mathsf{bmo}^c \cap \mathsf{bmo}^d}^2.$$

Hence

$$\|a\|_{\mathcal{BMO}^c} \leq \sqrt{2}\|a\|_{\mathsf{bmo}^c \cap \mathsf{bmo}^d}.$$

Passing to adjoints yields

$$\|a\|_{\mathsf{bmo}^r \cap \mathsf{bmo}^d} \leq \|a\|_{\mathcal{BMO}^r} \leq \sqrt{2}\|a\|_{\mathsf{bmo}^r \cap \mathsf{bmo}^d}.$$

These estimations show that the spaces $\mathcal{BMO}$ and bmo coincide. $\square$

We have the following duality:

Theorem 1.2.3. *We have* $(\mathsf{h}_1^c)^* = \mathsf{bmo}^c$ *with equivalent norms. More precisely,*

(i) *Every* $a \in \mathsf{bmo}^c$ *defines a continuous linear functional on* h_1^c *by*

$$\phi_a(x) = \tau(a^* x), \quad \forall x \in L_2(\mathcal{M}). \tag{1.2.1}$$

(ii) *Conversely, any* $\phi \in (\mathsf{h}_1^c)^*$ *is given as above by some* $a \in \mathsf{bmo}^c$.
Moreover

$$\|a\|_{\mathsf{bmo}^c} \leq \|\phi_a\|_{(h_1^c)^*} \leq \sqrt{2}\|a\|_{\mathsf{bmo}^c}.$$

Similarly, $(\mathsf{h}_1^r)^* = \mathsf{bmo}^r$ *and* $(\mathsf{h}_1)^* = \mathsf{bmo}$.

Remark 1.2.4. In the duality (1.2.1) we have identified an element $x \in L_2(\mathcal{M})$ with the martingale $(\mathcal{E}_n(x))_{n\geq 1}$. This martingale is in h_1^c and $\|x\|_{\mathsf{h}_1^c} \leq \|x\|_2$. Indeed, by the Hölder inequality, we have

$$\|x\|_{\mathsf{h}_1^c} = \|s_c(x)\|_1 \leq \|s_c(x)\|_2 = \|x\|_2,$$

where the last equality comes from the trace preserving property of conditional expectations and from the orthogonality in $L_2(\mathcal{M})$ of martingale difference sequences. As finite L_2-martingales are dense in h_1^c and in $L_2(\mathcal{M})$, we deduce that $L_2(\mathcal{M})$ is dense in h_1^c.

Proof. **Step 1:** We first show $\mathsf{bmo}^c \subset (h_1^c)^*$. This proof is similar to the corresponding one of the duality between $\mathcal{H}_1$ and $\mathcal{BMO}$ in [35]. Let $a \in \mathsf{bmo}^c$. Define ϕ_a by (1.2.1). We must show that ϕ_a induces a continuous linear functional on h_1^c.

Let x be a finite L_2-martingale. Then (recalling our identification between a martingale and its limit value if the latter exists)

$$\phi_a(x) = \sum_{n\geq 1} \tau(da_n^* dx_n).$$

Recall that

$$s_{c,n}(x) = \Big(\sum_{k=1}^n \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2} \quad \text{and} \quad s_c(x) = \Big(\sum_{k=1}^\infty \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2}.$$

By approximation we may assume that the $s_{c,n}(x)$'s are invertible elements in $\mathcal{M}$ for any $n \geq 1$.
Then by the Cauchy-Schwarz inequality and the tracial property of τ we have

$$\begin{aligned}
|\phi_a(x)| &= \Big|\sum_{n\geq 1} \tau(s_{c,n}(x)^{1/2}\, da_n^*\, dx_n\, s_{c,n}(x)^{-1/2})\Big| \\
&\leq \Big[\tau\Big(\sum_{n\geq 1} s_{c,n}(x)^{1/2}|da_n|^2 s_{c,n}(x)^{1/2}\Big)\Big]^{1/2} \\
&\quad \Big[\tau\Big(\sum_{n\geq 1} s_{c,n}(x)^{-1/2}|dx_n|^2 s_{c,n}(x)^{-1/2}\Big)\Big]^{1/2} \\
&= \Big[\tau\Big(\sum_{n\geq 1} s_{c,n}(x)|da_n|^2\Big)\Big]^{1/2} \Big[\tau\Big(\sum_{n\geq 1} s_{c,n}(x)^{-1}|dx_n|^2\Big)\Big]^{1/2} \\
&= : I \cdot II.
\end{aligned}$$

To estimate I we set $\theta_1 = s_{c,1}(x)$ and $\theta_n = s_{c,n}(x) - s_{c,n-1}(x)$ for $n \geq 1$. Then $\theta_n \in L_1(\mathcal{M}_{n-1})$ and $s_{c,n}(x) = \sum_{k=1}^{n} \theta_k$. Using the Abel summation and the modular property of conditional expectations, we find

$$\begin{aligned} I^2 &= \sum_{n\geq 1} \tau(s_{c,n}(x)|da_n|^2) = \sum_{n\geq 1}\sum_{k=1}^{n} \tau(\theta_k |da_n|^2) \\ &= \sum_{k\geq 1} \tau\Big(\theta_k \sum_{n\geq k} |da_n|^2\Big) = \sum_{k\geq 1} \tau\Big(\theta_k \mathcal{E}_{k-1}\Big(\sum_{n\geq k} |da_n|^2\Big)\Big) \\ &\leq \sum_{k\geq 1} \tau(\theta_k) \Big\| \mathcal{E}_{k-1}\Big(\sum_{n\geq k} |da_n|^2\Big)\Big\|_\infty \\ &\leq \|x\|_{\mathsf{h}_1^c} \|a\|^2_{\mathsf{bmo}^c}. \end{aligned}$$

To deal with II first note that

$$\tau\big[(s_{c,n}(x)^2 - s_{c,n-1}(x)^2) s_{c,n}(x)^{-1}\big] = \tau\big[(s_{c,n}(x) - s_{c,n-1}(x))(1 + s_{c,n-1}(x)s_{c,n}(x)^{-1})\big].$$

On the other hand, since $s_{c,n-1}(x)^2 \leq s_{c,n}(x)^2$, we find

$$\begin{aligned} \|s_{c,n-1}(x)s_{c,n}(x)^{-1}\|^2_\infty &= \|s_{c,n}(x)^{-1}s_{c,n-1}(x)^2 s_{c,n}(x)^{-1}\|_\infty \\ &\leq \|s_{c,n}(x)^{-1}s_{c,n}(x)^2 s_{c,n}(x)^{-1}\|_\infty = 1. \end{aligned}$$

As $\mathcal{E}_{n-1}(|dx_n|^2) = s_{c,n}(x)^2 - s_{c,n-1}(x)^2$ (with $s_{c,0}(x) = 0$) we have

$$\begin{aligned} II^2 &= \sum_{n\geq 1} \tau\big[\mathcal{E}_{n-1}(|dx_n|^2) s_{c,n}(x)^{-1}\big] \\ &= \sum_{n\geq 1} \tau\big[(s_{c,n}(x) - s_{c,n-1}(x))(1 + s_{c,n-1}(x)s_{c,n}(x)^{-1})\big] \\ &\leq \sum_{n\geq 1} \tau\big[s_{c,n}(x) - s_{c,n-1}(x)\big] \|1 + s_{c,n-1}(x)s_{c,n}(x)^{-1}\|_\infty \\ &\leq 2\tau\Big(\sum_{n\geq 1} s_{c,n}(x) - s_{c,n-1}(x)\Big) \\ &= 2\tau(s_c(x)) = 2\|x\|_{\mathsf{h}_1^c} \end{aligned}$$

Combining the preceding estimates on I and II, we obtain, for any finite L_2-martingale x,

$$|\phi_a(x)| \leq \sqrt{2}\|x\|_{\mathsf{h}_1^c}\|a\|_{\mathsf{bmo}^c}.$$

Therefore ϕ_a extends to an element of $(\mathsf{h}_1^c)^*$ of norm $\leq \sqrt{2}\|a\|_{\mathsf{bmo}^c}$.

Step 2: Let $\phi \in (\mathsf{h}_1^c)^*$ such that $\|\phi\|_{(\mathsf{h}_1^c)^*} \leq 1$. As $L_2(\mathcal{M}) \subset h_1^c$, ϕ induces a continuous functional $\widetilde{\phi}$ on $L_2(\mathcal{M})$. By the duality $(L_2(\mathcal{M}))^* = L_2(\mathcal{M})$, there exists $a \in L_2(\mathcal{M})$ such that

$$\widetilde{\phi}(x) = \tau(a^*x), \quad \forall x \in L_2(\mathcal{M}).$$

By the density of $L_2(\mathcal{M})$ in h_1^c (see Remark 1.2.4) we have

$$\|\phi\|_{(\mathsf{h}_1^c)^*} = \sup_{x\in L_2(\mathcal{M}), \|x\|_{\mathsf{h}_1^c}\leq 1} |\tau(a^*x)| \leq 1. \tag{1.2.2}$$

We will show that $a \in \mathsf{bmo}^c$. We want to estimate

$$\|a\|^2_{\mathsf{bmo}^c} = \max\Big(\|\mathcal{E}_1(a)\|^2_\infty \ , \ \sup_n \|\mathcal{E}_n|a - \mathcal{E}_n a|^2\|_\infty\Big).$$

Let $x \in L_1(\mathcal{M}_1), \|x\|_1 \leq 1$ be such that $\|\mathcal{E}_1(a)\|_\infty = |\tau(a^*x)|$. Then by (1.2.2) we have

$$\|\mathcal{E}_1(a)\|_\infty \leq \|x\|_{\mathsf{h}_1^c} = \|x\|_1 \leq 1.$$

On the other hand note that

$$\mathcal{E}_n|a - \mathcal{E}_n a|^2 = \mathcal{E}_n\Big(\sum_{k>n} |da_k|^2\Big) = \mathcal{E}_n\Big(\sum_{k>n} \mathcal{E}_{k-1}|da_k|^2\Big).$$

Fix $n \geq 1$. Let

$$z = s_c(a)^2 - s_{c,n}(a)^2 = \sum_{k>n} \mathcal{E}_{k-1}|da_k|^2.$$

We note that $z \in L_1(\mathcal{M})$ for $a \in L_2(\mathcal{M})$ and the orthogonality of martingale difference sequences in $L_2(\mathcal{M})$ gives

$$\|z\|_1 = \tau(z) = \tau\Big(\sum_{k>n} |da_k|^2\Big) \leq \tau\Big(\sum_{k\geq 1} |da_k|^2\Big) = \tau(|a|^2) = \|a\|_2^2.$$

Let $x \in L_1^+(\mathcal{M}_n), \|x\|_1 \leq 1$. Let y be the martingale defined as follows

$$dy_k = \begin{cases} 0 & \text{if } k \leq n \\ da_k x & \text{if } k > n \end{cases}.$$

By (1.2.2) we have

$$\tau(a^*y) \leq \|y\|_{\mathsf{h}_1^c}.$$

Since $x \in L_1^+(\mathcal{M}_n)$, we have

$$\begin{aligned} \tau(a^*y) &= \tau\Big(\sum_{k\geq 1} da_k^* dy_k\Big) = \tau\Big(\sum_{k>n} |da_k|^2 x\Big) \\ &= \tau\Big(\sum_{k>n} \mathcal{E}_{k-1}(|da_k|^2 x)\Big) \\ &= \tau\Big(\sum_{k>n} \mathcal{E}_{k-1}(|da_k|^2) x\Big) = \tau(zx). \end{aligned}$$

On the other hand, by the definition of y and the fact that $x \in L_1^+(\mathcal{M}_n)$, we find

$$\begin{aligned} s_c(y)^2 &= \sum_{k\geq 1} \mathcal{E}_{k-1}|dy_k|^2 = \sum_{k>n} \mathcal{E}_{k-1}|da_k x|^2 \\ &= \sum_{k>n} \mathcal{E}_{k-1}(x|da_k|^2 x) = \sum_{k>n} x\mathcal{E}_{k-1}(|da_k|^2)x = xzx. \end{aligned}$$

Thus

$$\|y\|_{\mathsf{h}_1^c} = \tau\Big((xzx)^{1/2}\Big).$$

Combining the preceding inequalities, we deduce

$$\tau(zx) \leq \tau((xzx)^{1/2}).$$

Since x is positive, using the Hölder inequality, we find

$$\begin{aligned} \tau\Big((xzx)^{1/2}\Big) &= \Big\|x^{1/2}(x^{1/2}zx^{1/2})x^{1/2}\Big\|_{1/2}^{1/2} \\ &\leq \Big(\|x^{1/2}\|_2 \|x^{1/2}zx^{1/2}\|_1 \|x^{1/2}\|_2\Big)^{1/2} = \tau(x)^{1/2}\tau(zx)^{1/2}. \end{aligned}$$

It then follows that

$$\tau(zx) \leq \tau(x),$$

whence

$$\tau(\mathcal{E}_n(z)x) = \tau(zx) \leq \tau(x) = \|x\|_1. \tag{1.2.3}$$

Taking the supremum over all $x \in L_1^+(\mathcal{M}_n)$ with $\|x\|_1 \leq 1$, we deduce $\|\mathcal{E}_n(z)\|_\infty \leq 1$. Therefore $a \in \mathsf{bmo}^c$ and $\|a\|_{\mathsf{bmo}^c} \leq 1$. This ends the proof of the duality $(\mathsf{h}_1^c)^* = \mathsf{bmo}^c$. Passing to adjoints yields the duality $(\mathsf{h}_1^r)^* = \mathsf{bmo}^r$.

Step 3: Since finite martingales are dense in each $\mathsf{h}_1^c, \mathsf{h}_1^r$ and h_1^d, the density property needed to apply the fact that the dual of a sum is the intersection of the duals holds. Thus it remains to determine the dual of h_1^d. Since h_1^d is a subspace of $\ell_1(L_1(\mathcal{M}))$, the Hahn-Banach theorem gives

$$(\mathsf{h}_1^d)^* = \frac{(\ell_1(L_1(\mathcal{M}))^*}{(\mathsf{h}_1^d)^\perp} = \frac{\ell_\infty(L_\infty(\mathcal{M}))}{(\mathsf{h}_1^d)^\perp}.$$

Let

$$P : \left\{ \begin{array}{lcc} \ell_\infty(L_\infty) & \longrightarrow & \mathsf{bmo}^d \\ (a_n)_{n\geq 1} & \longmapsto & (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n\geq 1} \end{array} \right. .$$

We claim that $\ker P = (\mathsf{h}_1^d)^\perp$. Indeed, for $a \in \ker P$ and $x \in \mathsf{h}_1^d$ we have

$$\begin{aligned} \langle dx, a\rangle &= \sum_{n\geq 1} \tau(dx_n^* a_n) = \sum_{n\geq 1} \Big[\tau(x_n^* a_n) - \tau(x_{n-1}^* a_n)\Big] \\ &= \sum_{n\geq 1} \Big[\tau(x_n^* \mathcal{E}_n(a_n)) - \tau(x_{n-1}^* \mathcal{E}_{n-1}(a_n))\Big] \\ &= \sum_{n\geq 1} \tau(dx_n^* \mathcal{E}_{n-1}(a_n)) \quad \text{for } \mathcal{E}_n(a_n) = \mathcal{E}_{n-1}(a_n) \\ &= 0. \end{aligned}$$

Conversely, if $a \in (\mathsf{h}_1^d)^\perp$ we fix $n \geq 1$ and define the martingale x by $dx_n = \mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)$ and $dx_m = 0$ if $m \neq n$. Since $a_n \in L_\infty(\mathcal{M})$ and τ is finite,

$$\sum_{m\geq 1} \|dx_m\|_1 = \|dx_n\|_1 \leq 2\|a_n\|_1 \leq 2\|a_n\|_\infty < \infty,$$

so $x \in \mathsf{h}_1^d$. Hence

$$\begin{aligned} 0 &= \langle dx, a\rangle = \tau\big[(\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))^* a_n\big] \\ &= \tau(\mathcal{E}_n(a_n)^* \mathcal{E}_n(a_n)) - \tau(\mathcal{E}_{n-1}(a_n)^* \mathcal{E}_{n-1}(a_n)) \\ &= \tau|\mathcal{E}_n a_n - \mathcal{E}_{n-1} a_n|^2; \end{aligned}$$

whence $\mathcal{E}_n(a_n) = \mathcal{E}_{n-1}(a_n)$. Thus we deduce that $a \in \ker P$. Therefore, our claim is proved. It then follows that $(\mathsf{h}_1^d)^* = \mathsf{bmo}^d$. Hence, the proof of the theorem is complete. □

We can now prove the reverse inclusion of Theorem 1.2.1.

Proof of Theorem 1.2.1. By the discussion following Theorem 1.2.1, we already know $\mathsf{h}_1 \subset \mathcal{H}_1$. To prove the reverse inclusion, we use duality. It then suffices to show $(\mathsf{h}_1)^* \subset (\mathcal{H}_1)^*$. To this end, by Theorem 1.2.3 and the duality theorem of [35], we must show $\mathsf{bmo} \subset \mathcal{BMO}$. This result is stated in Proposition 1.2.2 , with the equivalence constant $\sqrt{2}$. Combining the estimation of Theorem 1.2.3 and Proposition 1.2.2 with the appendix of [35], we obtain for any $a \in (\mathsf{h}_1)^*$

$$\|a\|_{(\mathcal{H}_1)^*} \leq \sqrt{2}\,\|a\|_{\mathcal{BMO}} \leq 2\|a\|_{\mathsf{bmo}} \leq 2\|a\|_{(\mathsf{h}_1)^*}$$

and

$$\|a\|_{(\mathsf{h}_1)^*} \leq \sqrt{2}\|a\|_{\mathsf{bmo}} \leq \sqrt{2}\|a\|_{\mathcal{BMO}} \leq \sqrt{6}\|a\|_{(\mathcal{H}_1)^*}.$$

□

Remark 1.2.5. Combining Proposition 1.2.2 and the duality results, we also obtain

$$\mathcal{H}_1^c = \mathsf{h}_1^c + \mathsf{h}_1^d \quad \text{and} \quad \mathcal{H}_1^r = \mathsf{h}_1^r + \mathsf{h}_1^d.$$

1.3 A description of the dual of h_p for $1 < p < 2$

In this section we extend the duality theorem in the previous section to the case $1 < p < 2$. Namely, we will describe the dual of h_p for $1 < p < 2$. The arguments are similar to those for $p = 1$. The situation becomes, however, a little more complicated since the noncommutative Doob maximal inequality is now involved. On the other hand, the proof of the duality theorem for $1 < p < 2$ is also slightly harder than that in the case $p = 1$. This partly explains why we have decided to first consider the case $p = 1$.

Let us recall the definition of the spaces $L_p(\mathcal{M};\ell_\infty), 1 \leq p \leq \infty$. A sequence $(x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if $(x_n)_{n\geq 1}$ admits a factorization $x_n = ay_nb$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_n)_{n\geq 1} \in \ell_\infty(L_\infty(\mathcal{M}))$. The norm of $(x_n)_{n\geq 1}$ is then defined as

$$\|(x_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_n = ay_nb} \|a\|_{2p} \sup_{n\geq 1} \|y_n\|_\infty \|b\|_{2p}.$$

One can check that $(L_p(\mathcal{M};\ell_\infty), \|\ \|_{L_p(\mathcal{M};\ell_\infty)})$ is a Banach space. It is proved in [20] and [26] that if $(x_n)_{n\geq 1}$ is a positive sequence in $L_p(\mathcal{M};\ell_\infty)$, then

$$\|(x_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_\infty)} = \sup\Big\{\sum_{n\geq 1} \tau(x_ny_n) : y_n \in L_{p'}^+(\mathcal{M}) \text{ and } \Big\|\sum_{n\geq 1} y_n\Big\|_{p'} \leq 1\Big\}. \quad (1.3.1)$$

The norm of $L_p(\mathcal{M};\ell_\infty)$ will be denoted by $\|\sup_n^+ x_n\|_p$. We should warn the reader that $\|\sup_n^+ x_n\|_p$ is just a notation since $\sup_n x_n$ does not take any sense in the noncommutative setting.

Now let $2 < q \leq \infty$. We define the space

$$L_q^c\mathsf{mo}(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : \|{\sup_{n\geq 1}}^+ \mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_{q/2} < \infty\}$$

equipped with the norm

$$\|a\|_{L_q^c\mathsf{mo}(\mathcal{M})} = \max\Big(\|\mathcal{E}_1(a)\|_q\ ,\ \Big(\|{\sup_{n\geq 1}}^+ \mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_{q/2}\Big)^{1/2}\Big).$$

Then $(L_q^c\mathsf{mo}(\mathcal{M}), \|\cdot\|_{L_q^c\mathsf{mo}(\mathcal{M})})$ is a Banach space. Similarly, we set

$$L_q^r\mathsf{mo}(\mathcal{M}) = \{a : a^* \in L_q^c\mathsf{mo}(\mathcal{M})\}$$

equipped with the norm

$$\|a\|_{L_q^r\mathsf{mo}(\mathcal{M})} = \|a^*\|_{L_q^c\mathsf{mo}(\mathcal{M})}.$$

Note that if $q = \infty$, then $L_\infty^c\mathsf{mo} = \mathsf{bmo}^c$ and $L_\infty^r\mathsf{mo} = \mathsf{bmo}^r$. For convenience we denote $L_q^c\mathsf{mo}(\mathcal{M}), L_q^r\mathsf{mo}(\mathcal{M})$ respectively by $L_q^c\mathsf{mo}, L_q^r\mathsf{mo}$.

The following duality holds:

Theorem 1.3.1. *Let $1 \leq p < 2$ and q be the index conjugate to p. Then $(\mathsf{h}_p^c)^* = L_q^c\mathsf{mo}$ with equivalent norms.*
More precisely,

(i) *Every $a \in L_q^c\mathsf{mo}$ defines a continuous linear functional on h_p^c by*

$$\phi_a(x) = \tau(a^*x), \quad \forall x \in L_2(\mathcal{M}). \tag{1.3.2}$$

(ii) *Conversely, any $\phi \in (\mathsf{h}_p^c)^*$ is given as above by some $a \in L_q^c\mathsf{mo}$.*
Moreover

$$\lambda_p^{-1/2}\|a\|_{L_q^c\mathsf{mo}} \leq \|\phi_a\|_{(\mathsf{h}_p^c)^*} \leq \sqrt{2}\,\|a\|_{L_q^c\mathsf{mo}} \tag{1.3.3}$$

where $\lambda_p > 0$ is a constant depending only on p and $\lambda_p = O(1)$ as $p \to 1$, $\lambda_p \leq C(2-p)^{-2}$ as $p \to 2$.

Similarly, we have $(\mathsf{h}_p^r)^* = L_q^r\mathsf{mo}$, and $(\mathsf{h}_p)^* = L_q^c\mathsf{mo} \cap L_q^r\mathsf{mo} \cap \mathsf{h}_q^d$.

Proof. We show only the duality equality $(\mathsf{h}_p^c)^* = L_q^c\mathsf{mo}$. To this end, we will adapt the proof of the corresponding duality result for $\mathcal{H}_p^c$ in [24] for the first step. The second one is adapted from the proof of Theorem 1.2.3.

Step 1: Let $a \in L_q^c\mathsf{mo}$ and x be a finite L_2-martingale such that $\|x\|_{\mathsf{h}_p^c} \leq 1$. Let s be the index conjugate to $\frac{q}{2}$. We consider

$$\widetilde{s}_{c,n}(x) = \Big(\sum_{k=1}^n \mathcal{E}_{k-1}|dx_k|^2\Big)^{p/2s} \quad \text{and} \quad \widetilde{s}_c(x) = \Big(\sum_{k=1}^\infty \mathcal{E}_{k-1}|dx_k|^2\Big)^{p/2s}.$$

Then $\widetilde{s}_{c,n}(x) \in L_s(\mathcal{M}_n)$ and by approximation we may assume that the $\widetilde{s}_{c,n}(x)$'s are invertible. By the arguments in the proof of the duality between h_1^c and bmo^c in Theorem 1.2.3 we have

$$\begin{aligned} |\phi_a(x)| &\leq \Big[\tau\Big(\textstyle\sum_{n\geq 1}\widetilde{s}_{c,n}(x)|da_n|^2\Big)\Big]^{1/2}\Big[\tau\Big(\textstyle\sum_{n\geq 1}\widetilde{s}_{c,n}(x)^{-1/2}|dx_n|^2\widetilde{s}_{c,n}(x)^{-1/2}\Big)\Big]^{1/2} \\ &=: I \cdot II. \end{aligned}$$

To estimate I we set again

$$\begin{cases} \theta_1 = \widetilde{s}_{c,1}(x) \\ \theta_n = \widetilde{s}_{c,n}(x) - \widetilde{s}_{c,n-1}(x), \quad \forall n \geq 2. \end{cases}$$

Then $\theta_n \in L_s(\mathcal{M}_{n-1})$, $\theta_n \geq 0$ and $\widetilde{s}_{c,n}(x) = \sum_{k=1}^n \theta_k$. Thus

$$\Big\|\sum_{k=1}^\infty \theta_k\Big\|_s = \|\widetilde{s}_c(x)\|_s = \|x\|_{h_p^c}^{p/s} \leq 1.$$

By (1.3.1), we have

$$\begin{aligned} I^2 &= \sum_{k\geq 1} \tau\Big(\theta_k \sum_{n\geq k} |da_n|^2\Big) \\ &= \sum_{k\geq 1} \tau\Big(\theta_k \mathcal{E}_{k-1}\Big(\sum_{n\geq k} |da_n|^2\Big)\Big) \\ &= \sum_{k\geq 1} \tau(\theta_k \mathcal{E}_{k-1}(|a-a_{k-1}|^2)) \\ &\leq \|{\sup_{k\geq 1}}^+ \mathcal{E}_k(|a-a_k|^2)\|_{q/2} = \|a\|^2_{L_q^c\mathsf{mo}}. \end{aligned}$$

To estimate the second term, let $\alpha = 2/p \in (1,2]$ and notice that

$$1-\alpha = 1-\frac{2}{p} = 1-2+\frac{2}{q} = -\frac{1}{s}.$$

For fixed n, we define $y = \widetilde{s}_{c,n-1}(x)^s$ and $z = \widetilde{s}_{c,n}(x)^s$. Since $p/2 \leq 1$, we have

$$y = \Big(\sum_{k=1}^{n-1} \mathcal{E}_{k-1}|dx_k|^2\Big)^{p/2} \leq \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\Big)^{p/2} = z.$$

Note that

$$z^{\frac{1-\alpha}{2}} = z^{-\frac{1}{2s}} = \widetilde{s}_{c,n}(x)^{-\frac{1}{2}}.$$

Applying Lemma 4.1 of [20], we find

$$\begin{aligned} \tau(\widetilde{s}_{c,n}(x)^{-1/2}\mathcal{E}_{n-1}|dx_n|^2\widetilde{s}_{c,n}(x)^{-1/2}) &= \tau(z^{\frac{1-\alpha}{2}}(z^\alpha - y^\alpha)z^{\frac{1-\alpha}{2}}) \\ &\leq 2\tau(z-y) \\ &= 2\tau(\widetilde{s}_{c,n}(x)^s - \widetilde{s}_{c,n-1}(x)^s). \end{aligned}$$

Therefore

$$\begin{aligned} II^2 &\leq 2\sum_{n\geq 1} \tau[\widetilde{s}_{c,n}(x)^s - \widetilde{s}_{c,n-1}(x)^s] \\ &= 2\tau[(\widetilde{s}_c(x)^s] \\ &= 2\tau\Big[\Big(\sum_{n\geq 1} \mathcal{E}_{n-1}|dx_n|^2\Big)^{p/2}\Big] \\ &= 2\|x\|^p_{\mathsf{h}_p^c} \leq 2. \end{aligned}$$

Combining the precedent estimations we deduce that for any finite L_2-martingale x

$$|\phi_a(x)| \leq \sqrt{2}\,\|a\|_{L_q^c\mathsf{mo}}\|x\|_{\mathsf{h}_p^c}.$$

Thus ϕ_a extends to an element of $(\mathsf{h}_p^c)^*$ with norm $\leq \sqrt{2}\,\|a\|_{L_q^c\mathsf{mo}}$.

Step 2: Let $\phi \in (\mathsf{h}_p^c)^*$ such that $\|\phi\|_{(\mathsf{h}_p^c)^*} \leq 1$. As $L_2(\mathcal{M}) \subset h_p^c$, ϕ induces a continuous functional $\widetilde{\phi}$ on $L_2(\mathcal{M})$. Thus there exists $a \in L_2(\mathcal{M})$ such that

$$\widetilde{\phi}(x) = \tau(a^*x), \quad \forall x \in L_2(\mathcal{M}).$$

By the density of $L_2(\mathcal{M})$ in h_p^c we have

$$\|\phi\|_{(\mathsf{h}_p^c)^*} = \sup_{x\in L_2(\mathcal{M}), \|x\|_{\mathsf{h}_p^c}\leq 1} |\tau(a^*x)| \leq 1. \tag{1.3.4}$$

We want to estimate

$$\|a\|^2_{L^c_q\mathsf{mo}} = \max\Big(\|\mathcal{E}_1(a)\|_q^2\ ,\ \Big\|\sup_{n\geq 1}{}^+\mathcal{E}_n\Big(\sum_{k>n}|da_k|^2\Big)\Big\|_{q/2}\Big).$$

Let $x \in L_p(\mathcal{M}_1), \|x\|_p \leq 1$ be such that $\|\mathcal{E}_1(a)\|_q = |\tau(a^*x)|$. Then by (1.3.4) we have

$$\|\mathcal{E}_1(a)\|_q \leq \|x\|_{\mathsf{h}^c_p} = \|x\|_p \leq 1.$$

On the other hand for each $n \geq 1$ we set

$$z_n = s_c(a)^2 - s_{c,n}(a)^2 = \sum_{k>n}\mathcal{E}_{k-1}|da_k|^2.$$

Then by (1.3.1) and the dual form of Junge's noncommutative Doob maximal inequality, we find (recalling that s is the conjugate index of $q/2$)

$$\begin{aligned}
&\|\sup_{n\geq 1}{}^+\mathcal{E}_n(z_n)\|_{q/2}\\
= \ & \sup\Big\{\sum_{n\geq 1}\tau(\mathcal{E}_n(z_n)b_n) : b_n \in L_s^+(\mathcal{M}) \text{ and } \Big\|\sum_{n\geq 1}b_n\Big\|_s \leq 1\Big\}\\
\leq \ & \lambda_s \sup\Big\{\sum_{n\geq 1}\tau(\mathcal{E}_n(z_n)b_n) : b_n \in L_s^+(\mathcal{M}_n) \text{ and } \Big\|\sum_{n\geq 1}b_n\Big\|_s \leq 1\Big\}.
\end{aligned}$$

Note that $\lambda_s = O(1)$ as s close to 1, so λ_s remains bounded as $q \to \infty$, i.e, as $p \to 1$. On the other hand, $\lambda_s \approx s^2$ as $s \to \infty$, i.e, as $p \to 2$.

Let $(b_n)_{n\geq 1}$ be a sequence in $L_s^+(\mathcal{M}_n)$ such that $\Big\|\sum_{n\geq 1}b_n\Big\|_s \leq 1$. Let y be the martingale defined as follows

$$dy_k = da_k\Big(\sum_{k>n}b_n\Big), \quad \forall k \geq 1.$$

By (1.3.4) we have

$$\tau(a^*y) \leq \|y\|_{\mathsf{h}^c_p}.$$

Since $b_n \in L_s^+(\mathcal{M}_n)$ for any $n \geq 1$, we have

$$\begin{aligned}
\tau(a^*y) &= \tau\Big(\sum_{k\geq 1}da_k^*dy_k\Big) = \tau\Big(\sum_{k\geq 1}|da_k|^2\Big(\sum_{k>n}b_n\Big)\Big)\\
&= \sum_{n\geq 1}\sum_{k>n}\tau(|da_k|^2b_n) = \sum_{n\geq 1}\sum_{k>n}\tau(\mathcal{E}_{k-1}|da_k|^2b_n)\\
&= \sum_{n\geq 1}\tau(z_nb_n)\\
&= \sum_{n\geq 1}\tau(\mathcal{E}_n(z_n)b_n).
\end{aligned}$$

On the other hand, by the definition of y and the fact that $b_n \in L_1^+(\mathcal{M}_n)$, we find

$$\begin{aligned}
s_c(y)^2 &= \sum_{k\geq 1}\mathcal{E}_{k-1}|dy_k|^2 = \sum_{k\geq 1}\mathcal{E}_{k-1}\Big|da_k\Big(\sum_{k>n}b_n\Big)\Big|^2\\
&= \sum_{k\geq 1}\mathcal{E}_{k-1}\Big[\Big(\sum_{k>n}b_n\Big)|da_k|^2\Big(\sum_{k>n}b_n\Big)\Big] = \sum_{k\geq 1}\sum_{n,m<k}b_n\mathcal{E}_{k-1}(|da_k|^2)b_m\\
&= \sum_{n,m\geq 1}b_nz_{\max(n,m)}b_m.
\end{aligned}$$

We consider the tensor product $\mathcal{N} = \mathcal{M}\overline{\otimes}B(\ell_2)$, equipped with the trace $\tau \otimes \mathrm{tr}$, where tr denote the usual trace on $B(\ell_2)$. Note that

$$\begin{aligned}
&\begin{bmatrix} b_1^{1/2} & b_2^{1/2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{bmatrix} \Big[b_n^{1/2} z_{\max(n,m)} b_m^{1/2}\Big]_{n,m\geq 1} \begin{bmatrix} b_1^{1/2} & 0 & \dots \\ b_2^{1/2} & 0 & \dots \\ \vdots & \vdots & \end{bmatrix} \\
= &\begin{bmatrix} \sum_{n,m\geq 1} b_n z_{\max(n,m)} b_m & 0 & \dots \\ 0 & \dots & \dots \\ \vdots & & \end{bmatrix}.
\end{aligned}$$

We claim that the matrix $Z = \Big[b_n^{1/2} z_{\max(n,m)} b_m^{1/2}\Big]_{n,m\geq 1}$ is positive. Indeed, we suppose that $\mathcal{M}$ acts on the Hilbert space H and we denote by $\langle \cdot, \cdot \rangle$ the associated scalar product. For $\xi = (\xi_n)_{n\geq 1} \in \ell_2(H)$, we have

$$\begin{aligned}
\langle Z\xi, \xi\rangle_{\ell_2(H)} &= \sum_{n,m\geq 1} \langle (b_n^{1/2} z_{\max(n,m)} b_m^{1/2})\xi_m, \xi_n\rangle \\
&= \sum_{n,m\geq 1} \langle z_{\max(n,m)} (b_m^{1/2}\xi_m), b_n^{1/2}\xi_n\rangle,
\end{aligned}$$

where the last equality comes from the positivity of the b_n's. Then the definition of z_n gives

$$\begin{aligned}
\langle Z\xi, \xi\rangle_{\ell_2(H)} &= \sum_{n,m\geq 1} \langle \Big(\sum_{k>\max(n,m)} \mathcal{E}_{k-1}|da_k|^2\Big) (b_m^{1/2}\xi_m), b_n^{1/2}\xi_n\rangle \\
&= \sum_{k\geq 1} \langle \mathcal{E}_{k-1}|da_k|^2 \Big(\sum_{m<k} b_m^{1/2}\xi_m\Big), \Big(\sum_{n<k} b_n^{1/2}\xi_n\Big)\rangle.
\end{aligned}$$

The positivity of the conditional expectation implies that each term of the latter sum is non-negative. Thus, we obtain

$$\langle Z\xi, \xi\rangle_{\ell_2(H)} \geq 0, \quad \forall \xi \in \ell_2(H),$$

which proves our claim. Hence

$$\|Z\|_{L_1(\mathcal{N})} = \tau \otimes \mathrm{tr}(Z) = \sum_{n\geq 1} \tau\Big(b_n^{1/2} z_n b_n^{1/2}\Big).$$

Since $\frac{2}{p} = \frac{1}{2s} + 1 + \frac{1}{2s}$, by the Hölder inequality we have

$$\begin{aligned}
&\Big\| \sum_{n,m\geq 1} b_n z_{\max(n,m)} b_m \Big\|_{p/2} = \Big\| \Big(\sum_{n,m\geq 1} b_n z_{\max(n,m)} b_m \Big) \otimes e_{1,1} \Big\|_{L_{p/2}(\mathcal{N})} \\
\leq &\left\| \begin{bmatrix} b_1^{1/2} & b_2^{1/2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{bmatrix} \right\|_{L_{2s}(\mathcal{N})} \Big\| \Big[b_n^{1/2} z_{\max(n,m)} b_m^{1/2}\Big]_{n,m\geq 1} \Big\|_{L_1(\mathcal{N})} \left\| \begin{bmatrix} b_1^{1/2} & 0 & \dots \\ b_2^{1/2} & 0 & \dots \\ \vdots & \vdots & \end{bmatrix} \right\|_{L_{2s}(\mathcal{N})} \\
= &\Big\| \sum_{n\geq 1} b_n \Big\|_s^{1/2} \Big[\sum_{n\geq 1} \tau\Big(b_n^{1/2} z_n b_n^{1/2}\Big) \Big] \Big\| \sum_{n\geq 1} b_n \Big\|_s^{1/2} \\
= &\Big[\sum_{n\geq 1} \tau(\mathcal{E}_n(z_n) b_n) \Big] \Big\| \sum_{n\geq 1} b_n \Big\|_s.
\end{aligned}$$

Thus

$$\|y\|_{\mathsf{h}_p^c} \leq \Big[\sum_{n\geq 1} \tau(\mathcal{E}_n(z_n) b_n) \Big]^{1/2} \Big\| \sum_{n\geq 1} b_n \Big\|_s^{1/2}.$$

Combining the preceding inequalities, we deduce

$$\sum_{n\geq 1} \tau(\mathcal{E}_n(z_n)b_n) \leq \Big\| \sum_{n\geq 1} b_n \Big\|_s \leq 1.$$

Therefore $a \in L_q^c\mathsf{mo}$ and $\|a\|_{L_q^c\mathsf{mo}} \leq \sqrt{\lambda_s}$. This ends the proof of the duality $(\mathsf{h}_p^c)^* = L_q^c\mathsf{mo}$. $\square$

Remark 1.3.2. Junge and Mei obtain in [21] the following inequality

$$\lambda_p^{-1}\|a\|_{L_q^c\mathsf{mo}} \leq \|\phi_a\|_{(\mathsf{h}_p^c)^*} \leq \sqrt{2}\,\|a\|_{L_q^c\mathsf{mo}}$$

where λ_p is the constant in (1.3.3). Note that our lower estimate is the square root of theirs, and yields a better estimation as $p \to 2$.

The dual space of $\mathcal{H}_p$ for $1 \leq p < 2$ is described in [24] as the space $L_q\mathcal{MO}$ (where q is the index conjugate of p) defined as follows. Let $2 < q \leq \infty$, we set

$$L_q^c\mathcal{MO}(\mathcal{M}) = \{a \in L_2(\mathcal{M}) : \|{\sup_{n\geq 1}}^+ \mathcal{E}_n(|a - \mathcal{E}_{n-1}(a)|^2)\|_{q/2} < \infty\},$$

equipped with the norm

$$\|a\|_{L_q^c\mathcal{MO}(\mathcal{M})} = \Big(\|{\sup_{n\geq 1}}^+ \mathcal{E}_n(|a - \mathcal{E}_{n-1}(a)|^2)\|_{q/2}\Big)^{1/2}.$$

Similarly, we define

$$L_q^r\mathcal{MO}(\mathcal{M}) = \{a : a^* \in L_q^c\mathcal{MO}(\mathcal{M})\},$$

equipped with the norm

$$\|a\|_{L_q^r\mathcal{MO}(\mathcal{M})} = \|a^*\|_{L_q^c\mathcal{MO}(\mathcal{M})}.$$

Finally, we set

$$L_q\mathcal{MO}(\mathcal{M}) = L_q^c\mathcal{MO}(\mathcal{M}) \cap L_q^r\mathcal{MO}(\mathcal{M}),$$

equipped with the intersection norm

$$\|x\|_{L_q\mathcal{MO}(\mathcal{M})} = \max\big(\|x\|_{L_q^c\mathcal{MO}(\mathcal{M})}\ ,\ \|x\|_{L_q^r\mathcal{MO}(\mathcal{M})}\big).$$

Note that if $q = \infty$, these spaces coincide with the $\mathcal{BMO}$ spaces. For convenience we denote $L_q^c\mathcal{MO}(\mathcal{M}), L_q^r\mathcal{MO}(\mathcal{M}), L_q\mathcal{MO}(\mathcal{M})$ respectively by $L_q^c\mathcal{MO}, L_q^r\mathcal{MO}, L_q\mathcal{MO}$. Theorem 4.1 of [24] establishes the duality $(\mathcal{H}_p^c)^* = L_q^c\mathcal{MO}$. Moreover, for any $a \in L_q^c\mathcal{MO}$,

$$\lambda_p^{-1}\|a\|_{L_q^c\mathcal{MO}} \leq \|\phi_a\|_{(\mathcal{H}_p^c)^*} \leq \sqrt{2}\,\|a\|_{L_q^c\mathcal{MO}}$$

where λ_p is the constant in (1.3.3).

Remark 1.3.3. The method used in the second step of the previous proof can be adapted to the duality between $\mathcal{H}_p^c$ and $L_q^c\mathcal{MO}$, for $1 < p < 2$. This yields a better estimate of the constant λ_p given in [24]. More precisely, we obtain by this way a constant of order $(2-p)^{-1}$ as $p \to 2$ instead of $(2-p)^{-2}$.

Indeed, let $\phi \in (\mathcal{H}_p^c)^*$ such that $\|\phi\|_{(\mathcal{H}_p^c)^*} \leq 1$. There exists $a \in L_2(\mathcal{M})$ such that

$$\phi(x) = \tau(a^*x), \quad \forall x \in L_2(\mathcal{M}).$$

By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_p^c$ we have

$$\|\phi\|_{(\mathcal{H}_p^c)^*} = \sup_{x\in L_2(\mathcal{M}), \|x\|_{\mathcal{H}_p^c}\leq 1} |\tau(a^*x)| \leq 1.$$

In this case, we want to estimate

$$\|a\|^2_{L_q^c\mathcal{MO}} = \|{\sup_{n\geq 1}}^+ \mathcal{E}_n(|a-\mathcal{E}_{n-1}(a)|^2)\|_{q/2} = \Big\|{\sup_{n\geq 1}}^+ \mathcal{E}_n\Big(\sum_{k\geq n}|da_k|^2\Big)\Big\|_{q/2}.$$

The triangular inequality in $L_{q/2}(\mathcal{M};\ell_\infty)$ allows us to separate the estimation into two parts as follows

$$\|a\|^2_{L_q^c\mathcal{MO}} \approx \Big\|{\sup_{n\geq 1}}^+ \mathcal{E}_n\Big(\sum_{k>n}|da_k|^2\Big)\Big\|_{q/2} + \|{\sup_{n\geq 1}}^+ |da_n|^2\|_{q/2} =: I + II.$$

We adapt the second step of the preceding proof by setting $z_n = \sum_{k>n}|da_k|^2$ for each $n\geq 1$. It yields the following estimation of the first term

$$I \leq \lambda_s$$

where s is the index conjugate to $\frac{q}{2}$.

To estimate the diagonal term II, let $(b_n)_{n\geq 1}$ be a sequence in $L_s^+(\mathcal{M}_n)$ such that $\Big\|\sum_{n\geq 1} b_n\Big\|_s \leq 1$. Let y be the martingale defined as follows

$$dy_k = da_k b_k - \mathcal{E}_{k-1}(da_k b_k), \quad \forall k\geq 1.$$

We have

$$\tau(a^*y) \leq \|y\|_{\mathcal{H}_p^c}.$$

Since $(da_n)_{n\geq 1}$ is a martingale difference sequence, we have

$$\begin{aligned}\tau(a^*y) &= \sum_{n\geq 1}\tau(|da_n|^2 b_n) - \tau(da_n^*\mathcal{E}_{n-1}(da_n b_n))\\ &= \sum_{n\geq 1}\tau(|da_n|^2 b_n) - \tau(\mathcal{E}_{n-1}(da_n^*)da_n b_n)\\ &= \sum_{n\geq 1}\tau(|da_n|^2 b_n).\end{aligned}$$

On the other hand, the triangular inequality in $L_p(\mathcal{M};\ell_2^c)$ yields

$$\|y\|_{\mathcal{H}_p^c} = \|(dy_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)} \leq \|(da_n b_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)} + \|(\mathcal{E}_{n-1}(da_n b_n))_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)}$$

The noncommutative Stein inequality implies

$$\|(\mathcal{E}_{n-1}(da_n b_n))_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)} \leq \gamma_p\|(da_n b_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)}$$

with $\gamma_p \leq C\frac{p^2}{p-1}$ (see [24]). Then

$$\|y\|_{\mathcal{H}_p^c} \leq (1+\gamma_p)\|(da_n b_n)_{n\geq 1}\|_{L_p(\mathcal{M};\ell_2^c)}.$$

As before, by the Hölder inequality, we find

$$\begin{aligned}
&\big\|(da_nb_n)_{n\geq 1}\big\|^2_{L_p(\mathcal{M};\ell_2^c)} = \Big\|\sum_{n\geq 1} b_n|da_n|^2 b_n\Big\|_{p/2} \\
\leq\ & \Big\|\sum_{n\geq 1} b_n\Big\|_s^{1/2}\Big[\sum_{n\geq 1}\tau\Big(b_n^{1/2}|da_n|^2 b_n^{1/2}\Big)\Big]\Big\|\sum_{n\geq 1} b_n\Big\|_s^{1/2} \\
=\ & \Big[\sum_{n\geq 1}\tau(|da_n|^2 b_n)\Big]\Big\|\sum_{n\geq 1} b_n\Big\|_s.
\end{aligned}$$

Hence

$$\|y\|_{\mathcal{H}_p^c} \leq (1+\gamma_p)\Big\|\sum_{n\geq 1} b_n\Big\|_s^{1/2}\Big(\sum_{n\geq 1}\tau(|da_n|^2 b_n)\Big)^{1/2}.$$

Combining the preceding inequalities, we deduce

$$\sum_{n\geq 1}\tau(|da_n|^2 b_n) \leq C(p-1)^{-2}\Big\|\sum_{n\geq 1} b_n\Big\|_s.$$

Then

$$II = \big\|{\sup_{n\geq 1}}^+ |da_n|^2\big\|_{q/2} \leq C(p-1)^{-2}\lambda_s.$$

Finally, we obtain

$$\|a\|_{L_q^c\mathcal{MO}} \leq \lambda_s^{1/2}(1+C(p-1)^{-2})^{1/2}.$$

Since $\lambda_s \approx s^2$ as $s\to\infty$, i.e, as $p\to 2$, we have the announced estimation $\lambda_s^{1/2}(1+C(p-1)^{-2})^{1/2} \approx (2-p)^{-1}$ as $p\to 2$.

For $1 < p < \infty$, the noncommutative Burkholder-Gundy inequalities of [35] and the noncommutative Burkholder inequalities of [24] state respectively that $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ and $\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M})$ (with equivalent norms). Combining these results we obtain the equivalence of the norms $\mathcal{H}_p$ and h_p. This is stated in Proposition 6.2 of [41]. Here Theorem 1.3.1 allows us to compare the dual spaces of $\mathcal{H}_p$ and h_p for $1\leq p<2$. This dual approach gives another way to compare the spaces $\mathcal{H}_p$ and h_p for $1\leq p<2$, which improve the estimation of the constant κ_p below for $1<p<2$. Indeed, Randrianantoananina obtained $\kappa_p = O((p-1)^{-1})$ as $p\to 1$ and the following statement gives that κ_p remains bounded as $p\to 1$. For completeness, we also include Randrianantoanina's estimates.

Theorem 1.3.4. *Let $1<p<\infty$. There exist two constants $\kappa_p>0$ and $\nu_p>0$ (depending only on p) such that for any finite L_p-martingale x,*

$$\kappa_p^{-1}\|x\|_{\mathsf{h}_p} \leq \|x\|_{\mathcal{H}_p} \leq \nu_p\|x\|_{\mathsf{h}_p}.$$

Moreover

(i) $\kappa_p \approx 1$ *as* $p\to 1$*;*

(ii) $\kappa_p \leq Cp$ *for* $2\leq p<\infty$*;*

(iii) $\nu_p \approx 1$ *as* $p\to 1$*;*

(iv) $\nu_p \leq C\sqrt{p}$ *for* $2\leq p<\infty$*.*

Proof. Randrianantoanina stated the estimations $(ii), (iii), (iv)$ in [41] without giving the proof. For the sake of completness we give the proof of these three estimations.

(i) Here we adopt a dual approch. Let $1 < p < 2$ and q the index conjugate to p. Let $a \in (\mathsf{h}_p)^*$. Then the triangular inequality in $L_{q/2}(\mathcal{M}; \ell_\infty)$ gives

$$\begin{aligned}
\|a\|^2_{L^c_q\mathcal{MO}} &= \Big\|\sup_{n\geq 1}{}^+ \mathcal{E}_n\Big(\sum_{k\geq n} |da_k|^2\Big)\Big\|_{q/2} \\
&\leq \Big\|\sup_{n\geq 1}{}^+ \mathcal{E}_n\Big(\sum_{k> n} |da_k|^2\Big)\Big\|_{q/2} + \|\sup_{n\geq 1}{}^+ \mathcal{E}_n|da_n|^2\|_{q/2} \\
&\leq \|a\|^2_{L^c_q\mathsf{mo}} + \|(\mathcal{E}_n|da_n|^2)_n\|_{L_{q/2}(\mathcal{M},\ell_\infty)}
\end{aligned}$$

But for $1 \leq p \leq \infty$ we have the following contractive inclusion

$$\ell_p(L_p(\mathcal{M})) \subset L_p(\mathcal{M}; \ell_\infty).$$

Therefore

$$\begin{aligned}
\|(\mathcal{E}_n|da_n|^2)_n\|_{L_{q/2}(\mathcal{M};\ell_\infty)} &\leq \|(\mathcal{E}_n|da_n|^2)_n\|_{\ell_{q/2}(L_{q/2})} \\
&\leq \Big(\sum_{n\geq 1} \|da_n\|_q^q\Big)^{2/q} = \|a\|^2_{\mathsf{h}^d_q}.
\end{aligned}$$

Then

$$\begin{aligned}
2^{-1/2}\|a\|_{(\mathcal{H}_p)^*} &\leq \|a\|_{L_q\mathcal{MO}} \\
&\leq \sqrt{2}\max(\|a\|_{L^c_q\mathsf{mo}}, \|a\|_{L^r_q\mathsf{mo}}, \|a\|_{h^d_q}) \\
&\leq \sqrt{2\lambda_p}\, \|a\|_{(h_p)^*}
\end{aligned}$$

with $\lambda_p = O(1)$ as $p \to 1$, hence $\kappa_p \approx 1$ as $p \to 1$.

(ii) The dual version of the noncommutative Doob inequality in [20] gives that for $1 \leq p < \infty$ and for all finite sequences (a_n) of positive elements in $L_p(\mathcal{M})$:

$$\Big\|\sum_{n\geq 1} \mathcal{E}_{n-1}(a_n)\Big\|_p \leq c_p \Big\|\sum_{n\geq 1} a_n\Big\|_p$$

with $c_p \approx p^2$ as $p \to +\infty$. Applying this to $a_n = |dx_n|^2$ and $p/2$ we get

$$\begin{aligned}
\|x\|_{h^c_p} &= \Big\|\Big(\sum_{n\geq 1} \mathcal{E}_{n-1}|dx_n|^2\Big)^{1/2}\Big\|_p = \Big\|\sum_{n\geq 1} \mathcal{E}_{n-1}|dx_n|^2\Big\|^{1/2}_{p/2} \\
&\leq \sqrt{c_{p/2}}\Big\|\sum_{n\geq 1} |dx_n|^2\Big\|^{1/2}_{p/2} = \sqrt{c_{p/2}}\,\|x\|_{\mathcal{H}^c_p}.
\end{aligned}$$

Passing to adjoints we have $\|x\|_{h^r_p} \leq \sqrt{c_{p/2}}\,\|x\|_{\mathcal{H}^r_p}$ with $\sqrt{c_{p/2}} \approx p$ as $p \to \infty$.
On the other hand, we have for $2 \leq p \leq \infty$ and for any finite sequence (a_n) in $L_p(\mathcal{M})$

$$\Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} \leq \Big\|\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2}\Big\|_p.$$

Indeed, this is trivially true for $p = 2$ and $p = \infty$. Then complex interpolation yields the intermediate case $2 < p < \infty$.
It thus follows that $\|dx\|_{\ell_p(L_p)} \leq \|x\|_{\mathcal{H}^c_p}$.
Thus $\kappa_p \leq Cp$ for $2 \leq p < \infty$.

(iii) Adapting the discussion following Theorem 1.2.1 to the case $0 < p < 1$, we obtain this estimate.

(iv) Suppose $2 < p < \infty$ and $\|x\|_{\mathsf{h}_p} \leq 1$. We write

$$|dx_n|^2 = \mathcal{E}_{n-1}|dx_n|^2 + (|dx_n|^2 - \mathcal{E}_{n-1}|dx_n|^2) =: \mathcal{E}_{n-1}|dx_n|^2 + dy_n.$$

The noncommutative Burkholder inequality implies

$$\begin{aligned}\Big\|\sum_{n\geq 1} dy_n\Big\|_{p/2} &= \|y\|_{p/2} \\ &\leq \eta_{p/2}\Big[\Big(\sum_{n\geq 1}\|dy_n\|_{p/2}^{p/2}\Big)^{2/p} + \Big\|\Big(\sum_{n\geq 1}\mathcal{E}_{n-1}|dy_n|^2\Big)^{1/2}\Big\|_{p/2}\Big] =: \eta_{p/2}(I+II)\end{aligned}$$

with $\eta_{p/2} \leq Cp$ as $p \to \infty$ from the proof of Theorem 4.1 of [41]. In order to estimate I we use the triangular inequality in $\ell_{p/2}(L_{p/2})$ and contractivity of the conditional expectations:

$$I = \|dy\|_{\ell_{p/2}(L_{p/2})} \leq 2\Big(\sum_{n\geq 1}\||dx_n|^2\|_{p/2}^{p/2}\Big)^{2/p} = 2\Big(\sum_{n\geq 1}\|dx_n\|_p^p\Big)^{2/p} \leq 2.$$

As for the second term II we note that

$$\mathcal{E}_{n-1}|dy_n|^2 = \mathcal{E}_{n-1}|dx_n|^4 - (\mathcal{E}_{n-1}|dx_n|^2)^2 \leq \mathcal{E}_{n-1}|dx_n|^4.$$

Then Lemma 5.2 of [24] gives the following estimation

$$\begin{aligned}II &\leq \Big\|\Big(\sum_{n\geq 1}\mathcal{E}_{n-1}|dx_n|^4\Big)^{1/2}\Big\|_{p/2} = \Big\|\sum_{n\geq 1}\mathcal{E}_{n-1}|dx_n|^4\Big\|_{p/4}^{1/2} \\ &\leq \Big(\Big\|\sum_{n\geq 1}\mathcal{E}_{n-1}|dx_n|^2\Big\|_{p/2}^{(p-4)/(p-2)}\Big(\sum_{n\geq 1}\|dx_n\|_p^p\Big)^{2/(p-2)}\Big)^{1/2} \\ &\leq 1.\end{aligned}$$

Combining the preceding inequalities we obtain

$$\|x\|_{\mathcal{H}_p^c}^2 \leq 1 + 3\eta_{p/2} \leq Cp \text{ as } p \to \infty.$$

□

Remark 1.3.5. At the time of this writing, we do not know if the orders of growth of κ_p and ν_p for $2 < p < \infty$ are optimal.

Chapter 2

Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales

Introduction

Atomic decomposition plays a fundamental role in the classical martingale theory and harmonic analysis. For instance, atomic decomposition is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities both in martingale theory and harmonic analysis. Atoms for martingales are usually defined in terms of stopping times. Unfortunately, the concept of stopping times is, up to now, not well-defined in the generic noncommutative setting (there are some works on this topic, see [1] and references therein). We note, however, that atoms can be defined without help of stopping times. Let us recall this in classical martingale theory. Given a probability space $(\Omega, \mathcal{F}, \mu)$, let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing filtration of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$ and let $(\mathbb{E}_n)_{n\geq 1}$ denote the corresponding family of conditional expectations. An $\mathcal{F}$-measurable function $a \in L_2$ is said to be an *atom* if there exist $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

(i) $\mathbb{E}_n(a) = 0$;

(ii) $\{a \neq 0\} \subset A$;

(iii) $\|a\|_2 \leq \mu(A)^{-1/2}$.

Such atoms are called *simple atoms* by Weisz [50] and are extensively studied by him (see [49] and [50]). Let us point out that atomic decomposition was first introduced in harmonic analysis by Coifman [6]. It is Herz [17] who initiated atomic decomposition for martingale theory. Recall that we denote by $\mathcal{H}_1(\Omega)$ the space of martingales f with respect to $(\mathcal{F}_n)_{n\geq 1}$ such that the quadratic variation $S(f) = \Big(\sum_n |df_n|^2\Big)^{1/2}$ belongs to $L_1(\Omega)$, and by $\mathsf{h}_1(\Omega)$ the space of martingales f such that the conditioned quadratic variation $s(f) = \Big(\sum_n \mathbb{E}_{n-1}|df_n|^2\Big)^{1/2}$ belongs to $L_1(\Omega)$. We say that a martingale $f = (f_n)_{n\geq 1}$ is predictable in L_1 if there exists an adapted sequence $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative functions such that $|f_n| \leq \lambda_{n-1}$ for all $n \geq 1$ and such that $\sup_n \lambda_n \in L_1(\Omega)$. We denote by $\mathcal{P}_1(\Omega)$ the space of all predictable martingales. In a disguised form in the proof of Theorem A_∞ in [17], Herz establishes an atomic description of $\mathcal{P}_1(\Omega)$. Since

$\mathcal{P}_1(\Omega) = \mathcal{H}_1(\Omega)$ for regular martingales, this gives an atomic decomposition of $\mathcal{H}_1(\Omega)$ in the regular case. Such a decomposition is still valid in the general case but for $\mathsf{h}_1(\Omega)$ instead of $\mathcal{H}_1(\Omega)$, as shown by Weisz [49].

In this paper, we will present the noncommutative version of atoms and prove that atomic decomposition for the Hardy spaces of noncommutative martingales is valid for these atoms. Since there are two kinds of Hardy spaces, i.e., the column and row Hardy spaces in the noncommutative setting, we need to define the corresponding two type atoms. This is a main difference from the commutative case, but can be done by considering the right and left supports of martingales as being operators on Hilbert spaces. Roughly speaking, replacing the supports of atoms in the above (ii) by the right (resp. left) supports we obtain the concept of noncommutative right (resp. left) atoms, which are proved to be suitable for the column (resp. row) Hardy spaces. On the other hand, due to the noncommutativity some basic constructions based on stopping times for classical martingales are not valid in the noncommutative setting, our approach to the atomic decomposition for the conditioned Hardy spaces of noncommutative martingales is via the $\mathsf{h}_1 - \mathsf{bmo}$ duality. Recall that the duality equality $(\mathsf{h}_1)^* = \mathsf{bmo}$ was established independently in [21] and in Chapter 1. However, this method does not give an explicit atomic decomposition.

The other main result of this paper concerns the interpolation of the conditioned Hardy spaces h_p. Such kind of interpolation results involving Hardy spaces of noncommutative martingales first appear in Musat's paper [31] for the spaces $\mathcal{H}_p$. We will present an extension of these results to the conditioned case. Note that our method is much simpler and more elementary than Musat's arguments. It seems that even in the commutative case, our method is simpler than all existing approaches to the interpolation of Hardy spaces of martingales. The main idea is inspired by an equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$ introduced by Herz [18] in the commutative case. We translate this quasinorm to the noncommutative setting to obtain a new characterization of h_p, $0 < p \leq 2$, which is more convenient for interpolation. By this way we show that $(\mathsf{bmo}, \mathsf{h}_1)_{1/p} = \mathsf{h}_p$ for any $1 < p < \infty$.

The study of the Hardy spaces of noncommutative martingales $\mathcal{H}_p$ and h_p in the discrete case is the starting point for the development of an $\mathcal{H}_p$-theory for continuous time. In a forthcoming paper by Marius Junge and the third named author, it appears that the spaces h_p are much easier to be handled than $\mathcal{H}_p$. It seems that their use is unavoidable for problems on the spaces $\mathcal{H}_p$ at the continuous time.

The remainder of this paper is divided into four sections. In Section 1 we present some preliminaries and notation on the noncommutative L_p-spaces and various Hardy spaces of noncommutative martingales. The atomic decomposition of the conditioned Hardy space $\mathsf{h}_1(\mathcal{M})$ is presented in Section 2, from which we deduce the atomic decomposition of the Hardy space $\mathcal{H}_1(\mathcal{M})$ by Davis' decomposition. In Section 3 we define an equivalent quasinorm for $\mathsf{h}_p(\mathcal{M}), 0 < p \leq 2$, and discuss the description of the dual space of $\mathsf{h}_p(\mathcal{M}), 0 < p \leq 1$. Finally, using the results of Section 3, the interpolation results between bmo and h_1 are proved in Section 4.

Any notation and terminology not otherwise explained, are as used in [46] for theory of von Neumann algebras, and in [36] for noncommutative L_p-spaces. Also, we refer to a recent book by Xu [53] for an up-to-date exposition of theory of noncommutative martingales.

2.1 Preliminaries and notations

Throughout this paper, $\mathcal{M}$ will always denote a von Neumann algebra with a normal faithful normalized trace τ. For each $0 < p \leq \infty$, let $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ be the associated noncommutative L_p-spaces. We refer to [36] for more details and historical references on these spaces.

For $x \in L_p(\mathcal{M})$ we denote by $r(x)$ and $l(x)$ the right and left supports of x, respectively. Recall that if $x = u|x|$ is the polar decomposition of x, then $r(x) = u^*u$ and $l(x) = uu^*$. $r(x)$ (resp. $l(x)$) is also the least projection e such that $xe = x$ (resp. $ex = x$). If x is selfadjoint, $r(x) = l(x)$.

Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is w*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$.

A sequence $x = (x_n)$ in $L_1(\mathcal{M})$ is called a *noncommutative martingale* with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$.

If in addition, all x_n's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called an L_p-martingale. In this case we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called a bounded L_p-martingale.

Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $dx = (dx_n)$ is called the *martingale difference sequence* of x. x is called a *finite martingale* if there exists N such that $dx_n = 0$ for all $n \geq N$. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Let us now recall the definitions of the square functions and Hardy spaces for noncommutative martingales. Following [35], we introduce the column and row versions of square functions relative to a (finite) martingale $x = (x_n)$:

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

Let $1 \leq p < \infty$. Define $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) as the completion of all finite L_p-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$). The Hardy space of noncommutative martingales is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\},$$

where the infimum is taken over all $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$ such that $x = y + z$. For $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

The reason that $\mathcal{H}_p(\mathcal{M})$ is defined differently according to $1 \leq p < 2$ or $2 \leq p \leq \infty$ is presented in [35]. In that paper Pisier and Xu prove the noncommutative Burkholder-Gundy inequalities which imply that $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ with equivalent norms for $1 < p < \infty$.

We now consider the conditioned version of $\mathcal{H}_p$ developed in [24]. Let $x = (x_n)_{n\geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k|^2\Big)^{1/2};$$

and

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{1/2}.$$

These will be called the column and row conditioned square functions, respectively. Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi)norm $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). For $p = \infty$, we define $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) as the Banach space of the $L_\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology.

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{if } 0 < p < \infty,$$

and

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty \quad \text{if } p = \infty.$$

Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.

We define the conditioned version of martingale Hardy spaces as follows: If $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M})$$

equipped with the (quasi)norm

$$\|x\|_{\mathsf{h}_p} = \inf\big\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r}\big\},$$

where the infimum is taken over all $w \in \mathsf{h}_p^d(\mathcal{M}), y \in \mathsf{h}_p^c(\mathcal{M})$ and $z \in \mathsf{h}_p^r(\mathcal{M})$ such that $x = w + y + z$. For $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) \cap \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p} = \max\big\{\|x\|_{\mathsf{h}_p^d}, \|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}\big\}.$$

The noncommutative Burkholder inequalities proved in [24] state that

$$\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M}) \tag{2.1.1}$$

with equivalent norms for all $1 < p < \infty$.

In the sequel, $(\mathcal{M}_n)_{n\geq 1}$ will be a filtration of von Neumann subalgebras of $\mathcal{M}$. All martingales will be with respect to this filtration.

2.2 Atomic decompositions

Let us now introduce the concept of noncommutative atoms.

Definition 2.2.1. *$a \in L_2(\mathcal{M})$ is said to be a $(1,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that*

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{-1/2}$.

Replacing (ii) *by* (ii)′ $l(a) \leq e$, *we get the notion of a* $(1,2)_r$*-atom.*

Here, $(1,2)_c$-atoms and $(1,2)_r$-atoms are noncommutative analogues of $(1,2)$-atoms for classical martingales. In a later remark we will discuss the noncommutative analogue of $(p,2)$-atoms. These atoms satisfy the following useful estimates.

Proposition 2.2.2. *If a is a $(1,2)_c$-atom then*

$$\|a\|_{\mathcal{H}_1^c} \leq 1 \quad \text{and} \quad \|a\|_{\mathsf{h}_1^c} \leq 1.$$

The similar inequalities hold for $(1,2)_r$-atoms.

Proof. Let e be a projection associated with a satisfying (i) – (iii) of Definition 2.2.1. Let $a_k = \mathcal{E}_k(a)$. Observe that $a_k = 0$ for $k \leq n$, so $da_k = 0$ for $k \leq n$. For $k \geq n+1$ we have

$$\begin{array}{rclcl} e|da_k|^2 & = & [\mathcal{E}_k(ea^*) - \mathcal{E}_{k-1}(ea^*)]da_k & = & |da_k|^2 \\ & = & da_k^*[\mathcal{E}_k(ae) - \mathcal{E}_{k-1}(ae)] & = & |da_k|^2 e. \end{array}$$

This gives

$$e|da_k|^2 = |da_k|^2 = |da_k|^2 e$$

for any $k \geq 1$. Hence, we obtain

$$eS_c(a) = S_c(a) = S_c(a)e.$$

Consequently, the noncommutative Hölder inequality implies

$$\|a\|_{\mathcal{H}_1^c} = \tau[eS_c(a)] \leq \|S_c(a)\|_2 \|e\|_2 = \|a\|_2 \|e\|_2 \leq 1.$$

Since $e \in \mathcal{M}_n$, for $k \geq n+1$ we have

$$\begin{array}{rclcl} e\mathcal{E}_{k-1}(|da_k|^2) & = & \mathcal{E}_{k-1}(e|da_k|^2) & = & \mathcal{E}_{k-1}(|da_k|^2) \\ & = & \mathcal{E}_{k-1}(|da_k|^2 e) & = & \mathcal{E}_{k-1}(|da_k|^2)e. \end{array}$$

Thus, we deduce

$$\|a\|_{\mathsf{h}_1^c} \leq 1.$$

□

Now, atomic Hardy spaces are defined as follows.

Definition 2.2.3. *We define* $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ *as the Banach space of all* $x \in L_1(\mathcal{M})$ *which admit a decomposition*

$$x = \sum_k \lambda_k a_k$$

with for each k, a_k *a* $(1,2)_c$*-atom or an element in* $L_1(\mathcal{M}_1)$ *of norm* ≤ 1, *and* $\lambda_k \in \mathbb{C}$ *satisfying* $\sum_k |\lambda_k| < \infty$. *We equip this space with the norm*

$$\|x\|_{\mathsf{h}_1^{c,\mathrm{at}}} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of x *described above.*

Similarly, we define $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ *and* $\|\cdot\|_{\mathsf{h}_1^{r,\mathrm{at}}}$.

It is easy to see that $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ is a Banach space. By Proposition 2.2.2 we have the contractive inclusion $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_1^c(\mathcal{M})$. The following theorem shows that these two spaces coincide. That establishes the atomic decomposition of the conditioned Hardy space $\mathsf{h}_1^c(\mathcal{M})$. This is the main result of this section.

Theorem 2.2.4. *We have*

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}_1^c(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}_1^{c,\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1^c} \leq \|x\|_{\mathsf{h}_1^{c,\mathrm{at}}}.$$

Similarly, $\mathsf{h}_1^r(\mathcal{M}) = \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ *with the same equivalence constants.*

We will show the remaining inclusion $\mathsf{h}_1^c(\mathcal{M}) \subset \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ by duality. Recall that the dual space of $\mathsf{h}_1^c(\mathcal{M})$ is the space $\mathsf{bmo}^c(\mathcal{M})$ defined as follows (we refer to [21] and Chapter 1 for details). Let

$$\mathsf{bmo}^c(\mathcal{M}) = \big\{x \in L_2(\mathcal{M}) : \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty < \infty\big\}$$

and equip $\mathsf{bmo}^c(\mathcal{M})$ with the norm

$$\|x\|_{\mathsf{bmo}^c} = \max\Big(\|\mathcal{E}_1(x)\|_\infty\ ,\ \sup_{n\geq 1}\|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\Big).$$

This is a Banach space. Similarly, we define the row version $\mathsf{bmo}^r(\mathcal{M})$. Since $x_n = \mathcal{E}_n(x)$, we have

$$\mathcal{E}_n|x - x_n|^2 = \mathcal{E}_n|x|^2 - |x_n|^2 \leq \mathcal{E}_n|x|^2.$$

Thus the contractivity of the conditional expectation yields

$$\|x\|_{\mathsf{bmo}^c} \leq \|x\|_\infty. \tag{2.2.1}$$

We will describe the dual space of $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ as a noncommutative Lipschitz space defined as follows. We set

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

with

$$\|x\|_{\Lambda^c} = \max\Big(\|\mathcal{E}_1(x)\|_\infty\ ,\ \sup_{n\geq 1}\sup_{e\in\mathcal{P}_n} \tau(e)^{-1/2}\tau\big(e|x - x_n|^2\big)^{1/2}\Big),$$

where $\mathcal{P}_n$ denotes the lattice of projections of $\mathcal{M}_n$. Similarly, we define

$$\Lambda^r(\mathcal{M}) = \{x \in L^2(\mathcal{M}) : x^* \in \Lambda^c(\mathcal{M})\}$$

equipped with the norm

$$\|x\|_{\Lambda^r} = \|x^*\|_{\Lambda^c}.$$

The relation between Lipschitz space and bmo space can be stated as follows.

Proposition 2.2.5. *We have* $\mathsf{bmo}^c(\mathcal{M}) = \Lambda^c(\mathcal{M})$ *and* $\mathsf{bmo}^r(\mathcal{M}) = \Lambda^r(\mathcal{M})$ *isometrically.*

Proof. Let $x \in \mathsf{bmo}^c(\mathcal{M})$. It is obvious that by the noncommutative Hölder inequality we have, for all $n \geq 1$,

$$\sup_{e \in \mathcal{P}_n} \tau(e)^{-1/2} \tau\big(e|x - x_n|^2\big)^{1/2} \leq \|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}.$$

To prove the reverse inclusion, by duality we can write

$$\begin{aligned} \|\mathcal{E}_n|x - x_n|^2\|_\infty &= \sup_{\|y\|_1 \leq 1,\ y \in L_1^+(\mathcal{M}_n)} \big|\tau(y|x - x_n|^2)\big| \\ &= \sup_{e \in \mathcal{P}_n} \tau(e)^{-1} \tau(e|x - x_n|^2), \end{aligned}$$

where the last equality comes from the density of linear combinations of mutually disjoint projections in $L_1(\mathcal{M}_n)$. Thus $\|x\|_{\Lambda^c} = \|x\|_{\mathsf{bmo}^c}$, and the same holds for the row spaces. $\square$

We now turn to the duality between the conditioned atomic space $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ and the Lipschitz space $\Lambda^c(\mathcal{M})$.

Theorem 2.2.6. *We have* $h_1^{c,\mathrm{at}}(\mathcal{M})^* = \Lambda^c(\mathcal{M})$ *isometrically. More precisely,*

(i) *Every* $x \in \Lambda^c(\mathcal{M})$ *defines a continuous linear functional on* $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ *by*

$$\varphi_x(y) = \tau(x^* y), \quad \forall y \in L_2(\mathcal{M}). \tag{2.2.2}$$

(ii) *Conversely, each* $\varphi \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})^*$ *is given as* (2.2.2) *by some* $x \in \Lambda^c(\mathcal{M})$.

Similarly, $\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})^* = \Lambda^r(\mathcal{M})$ *isometrically.*

Remark 2.2.7. Remark that we have defined the duality bracket (2.2.2) for operators in $L_2(\mathcal{M})$. This is sufficient for $L_2(\mathcal{M})$ is dense in $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$. The latter density easily follows from the decomposition $L_2(\mathcal{M}) = L_2^0(\mathcal{M}) \oplus L_2(\mathcal{M}_1)$, where $L_2^0(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \mathcal{E}_1(x) = 0\}$.

Proof of Theorem 2.2.6. We first show $\Lambda^c(\mathcal{M}) \subset h_1^{c,\mathrm{at}}(\mathcal{M})^*$. In fact we will not need this inclusion for the proof of Theorem 2.2.4, however we include the proof for the sake of completeness. Let $x \in \Lambda^c(\mathcal{M})$. For any $(1,2)_c$-atom a associated with a projection e satisfying (i) – (iii) of Definition 2.2.1, by the noncommutative Hölder inequality we have

$$\begin{aligned} \big|\tau(x^* a)\big| &= \big|\tau((x - x_n)^* a e)\big| \\ &\leq \|e(x - x_n)^*\|_2 \|a\|_2 \\ &\leq \tau(e)^{-1/2} \big[\tau(e|x - x_n|^2)\big]^{1/2} \\ &\leq \|x\|_{\Lambda^c}. \end{aligned}$$

On the other hand, for any $a \in L_1(\mathcal{M}_1)$ with $\|a\|_1 \leq 1$ we have

$$|\tau(x^*a)| = |\tau(\mathcal{E}_1(x)^*a)| \leq \|\mathcal{E}_1(x)\|_\infty \|a\|_1 \leq \|x\|_{\Lambda^c}.$$

Thus, we deduce that

$$|\tau(x^*y)| \leq \|x\|_{\Lambda^c} \|y\|_{\mathsf{h}_1^{c,\mathrm{at}}}$$

for all $y \in L_2(\mathcal{M})$. Hence, φ_x extends to a continuous functional on $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ of norm less than or equal to $\|x\|_{\Lambda^c}$.

Conversely, let $\varphi \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})^*$. As explained in the previous remark, $L_2(\mathcal{M}) \subset \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ so by the Riesz representation theorem there exists $x \in L_2(\mathcal{M})$ such that

$$\varphi(y) = \tau(x^*y), \quad \forall y \in L_2(\mathcal{M}).$$

Fix $n \geq 1$ and let $e \in \mathcal{P}_n$. We set

$$y_e = \frac{(x - x_n)e}{\|(x - x_n)e\|_2 \tau(e)^{1/2}}.$$

It is clear that y_e is a $(1,2)_c$-atom with the associated projection e. Then

$$\|\varphi\| \geq |\varphi(y_e)| = |\tau((x - x_n)^* y_e)| = \frac{1}{\tau(e)^{1/2}} \big[\tau(e|x - x_n|^2)\big]^{1/2}.$$

On the other hand, let $y \in L_1(\mathcal{M}_1), \|y\|_1 \leq 1$ be such that $\|\mathcal{E}_1(x)\|_\infty = |\tau(x^*y)|$. Then $\|\mathcal{E}_1(x)\|_\infty \leq \|\varphi\|$. Combining these estimates we obtain $\|x\|_{\Lambda^c} \leq \|\varphi\|$. This ends the proof of the duality $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = \Lambda^c(\mathcal{M})$. Passing to adjoints yields the duality $(\mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}))^* = \Lambda^r(\mathcal{M})$. □

We can now prove the reverse inclusion of Theorem 2.2.4.

Proof of Theorem 2.2.4. By Proposition 2.2.2 we already know that $\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_1^c(\mathcal{M})$. Combining Proposition 2.2.5 and Theorem 2.2.6 we obtain that $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = \mathsf{bmo}^c(\mathcal{M})$ with equal norms. The duality between $\mathsf{h}_1^c(\mathcal{M})$ and $\mathsf{bmo}^c(\mathcal{M})$ proved in [21] and Chapter 1 then yields that $(\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}))^* = (\mathsf{h}_1^c(\mathcal{M}))^*$ with the following equivalence constants

$$\frac{1}{\sqrt{2}} \|\varphi_x\|_{(\mathsf{h}_1^c)^*} \leq \|x\|_{\mathsf{bmo}^c} = \|\varphi_x\|_{(\mathsf{h}_1^{c,\mathrm{at}})^*} \leq \|\varphi_x\|_{(\mathsf{h}_1^c)^*}.$$

This ends the proof of Theorem 2.2.4. □

We can generalize this decomposition to the whole space $\mathsf{h}_1(\mathcal{M})$. To this end we need the following definition.

Definition 2.2.8. *We set*

$$\mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_1^d(\mathcal{M}) + \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) + \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M}),$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_1^{\mathrm{at}}} = \inf\big\{\|w\|_{\mathsf{h}_1^d} + \|y\|_{\mathsf{h}_1^{c,\mathrm{at}}} + \|z\|_{\mathsf{h}_1^{r,\mathrm{at}}}\big\},$$

where the infimum is taken over all $w \in \mathsf{h}_1^d(\mathcal{M}), y \in \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$, and $z \in \mathsf{h}_1^{r,\mathrm{at}}(\mathcal{M})$ such that $x = w + y + z$.

Thus Theorem 2.2.4 clearly implies the following.

Theorem 2.2.9. *We have*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M}) \qquad \textit{with equivalent norms.}$$

More precisely, if $x \in \mathsf{h}_1(\mathcal{M})$

$$\frac{1}{\sqrt{2}}\|x\|_{\mathsf{h}_1^{\mathrm{at}}} \leq \|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathsf{h}_1^{\mathrm{at}}}.$$

The noncommutative Davis' decomposition presented in Chapter 1 states that $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})$. Thus Theorem 2.2.9 yields that $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1^{\mathrm{at}}(\mathcal{M})$, which means that we can decompose any martingale in $\mathcal{H}_1(\mathcal{M})$ in an atomic part and a diagonal part. This is the atomic decomposition for the Hardy space of noncommutative martingales.

2.3 An equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$

In the commutative case Herz described in [18] an equivalent quasinorm for $\mathsf{h}_p, 0 < p \leq 2$. This section is devoted to determining a noncommutative analogue of this. This characterization of h_p will be useful in the sequel. Indeed, this will imply an interpolation result in the next section. To define equivalent quasinorms of $\|\cdot\|_{\mathsf{h}_p^c}$ and $\|\cdot\|_{\mathsf{h}_p^r}$ for $0 < p \leq 2$ we introduce the index class W which consists of sequences $\{w_n\}_{n\in\mathbb{N}}$ such that $\{w_n^{2/p-1}\}_{n\in\mathbb{N}}$ is nondecreasing with each $w_n \in L_1^+(\mathcal{M}_n)$ invertible with bounded inverse and $\|w_n\|_1 \leq 1$.

For an L_2-martingale x we set

$$N_p^c(x) = \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}|^2\Big)\Big]^{1/2}$$

and

$$N_p^r(x) = \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}^*|^2\Big)\Big]^{1/2}.$$

We need the following well-known lemma, and include a proof for the convenience of the reader (see Lemma 1 of [47] for the case $f(t) = t^p$).

Lemma 2.3.1. *Let f be a function in $C^1(\mathbb{R}^+)$ and $x, y \in \mathcal{M}^+$. Then*

$$\tau(f(x+y) - f(x)) = \tau\Big(\int_0^1 f'(x+ty)ydt\Big).$$

Proof. Note that considering $f - f(0)$, we may assume that $f(0) = 0$. We set $\varphi_f(t) = \tau(f(x+ty))$, for $t \in [0,1]$. Then

$$\varphi_f'(t) = \tau(f'(x+ty)y), \quad \forall t \in [0,1]. \tag{2.3.1}$$

Indeed, the tracial property of τ implies this equality for $t = 0$ and $f(t) = t^n, n \in \mathbb{N}$, and we can extend this result for all f polynomials by linearity. A translation argument gives (2.3.1) for all f polynomials. Finally, we generalize for all f by approximation. Indeed, we can approximate f' by a sequence $(p_n)_{n\geq 1}$ of polynomials, uniformly on the compact set $K = [0, \|x\|_\infty + \|y\|_\infty]$. Then the sequence of polynomials (q_n) defined by $q_n(s) = \int_0^s p_n(t)dt$ for each $n \geq 1$ converges uniformly to f on K. Since (φ_{q_n}') converges to φ_f' uniformly on $[0,1]$ (by the derivation theorem), we get (2.3.1) by the finiteness of the trace.
Now writing $\varphi_f(1) - \varphi_f(0) = \int_0^1 \varphi_f'(t)dt$ we obtain the desired result. $\square$

Proposition 2.3.2. *For $0 < p \leq 2$ and $x \in L_2(\mathcal{M})$ we have*

$$\Big(\frac{p}{2}\Big)^{1/2} N_p^c(x) \leq \|x\|_{\mathsf{h}_p^c} \leq N_p^c(x). \tag{2.3.2}$$

A similar statement holds for $\mathsf{h}_p^r(\mathcal{M})$ and N_p^r.

Proof. Note that

$$\begin{aligned} N_p^c(x) &= \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}\mathcal{E}_n|dx_{n+1}|^2\Big)\Big]^{1/2} \\ &= \inf_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)\Big)\Big]^{1/2}. \end{aligned}$$

Let $x \in L_2(\mathcal{M})$ with $\|x\|_{\mathsf{h}_p^c} < 1$. By approximation we can assume that $x \in L_\infty(\mathcal{M})$ and $s_{c,n}(x)$ is invertible with bounded inverse for every $n \geq 1$. Then $\{s_{c,n+1}(x)^p\} \in W$; so

$$N_p^c(x) \leq \Big[\tau\Big(\sum_{n\geq 0} s_{c,n+1}(x)^{p-2}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)\Big)\Big]^{1/2}.$$

Applying Lemma 2.3.1 with $f(t) = t^{p/2}, x + y = s_{c,n+1}(x)^2$ and $x = s_{c,n}(x)^2$ we obtain

$$\begin{aligned} &\tau(s_{c,n+1}(x)^p - s_{c,n}(x)^p) = \\ &\tau\Big(\int_0^1 \frac{p}{2}[s_{c,n}(x)^2 + t(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)]^{\frac{p}{2}-1}[s_{c,n+1}(x)^2 - s_{c,n}(x)^2]dt\Big) \\ &\geq \frac{p}{2}\tau(s_{c,n+1}(x)^{p-2}(s_{c,n+1}(x)^2 - s_{c,n}(x)^2)), \end{aligned}$$

where we have used the fact that the operator function $a \mapsto a^{\frac{p}{2}-1}$ is nonincreasing for $-1 < \frac{p}{2} - 1 \leq 0$. Taking the sum over n leads to

$$N_p^c(x)^2 \leq \frac{2}{p}\tau(s_c(x)^p) = \frac{2}{p}.$$

We turn to the other estimate. Given $\{w_n\} \in W$ put

$$w^{2/p-1} = \lim_{n\to+\infty} w_n^{2/p-1} = \sup_n w_n^{2/p-1}.$$

It follows that $\{w_n^{1-2/p}\}$ decreases to $w^{1-2/p}$ and

$$\begin{aligned} \tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}|^2\Big) &\geq \tau\Big(w^{1-2/p}\sum_{n\geq 0}\mathcal{E}_n|dx_{n+1}|^2\Big) \\ &= \tau\Big(w^{1-2/p}s_c(x)^2\Big). \end{aligned}$$

Since $\frac{1}{p} = \frac{1}{2} + \frac{2-p}{2p}$ the Hölder inequality gives

$$\begin{aligned} \|s_c(x)\|_p &= \|w^{1/p-1/2}w^{1/2-1/p}s_c(x)\|_p \\ &\leq \|w^{1/p-1/2}\|_{2p/(2-p)}\|w^{1/2-1/p}s_c(x)\|_2 \\ &= \tau(w)^{1/p-1/2}\tau(w^{1-2/p}s_c(x)^2)^{1/2}. \end{aligned}$$

Now $\tau(w) \leq 1$; so we have

$$\|s_c(x)\|_p \leq \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/p}|dx_{n+1}|^2\Big)\Big]^{1/2}$$

for all $\{w_n\} \in W$. $\square$

Thus the quasinorm N_p^c is equivalent to $\|\cdot\|_{\mathsf{h}_p^c}$ on $L_2(\mathcal{M})$. So $\mathsf{h}_p^c(\mathcal{M})$ can also be defined as the completion of all finite L_2-martingales with respect to N_p^c for $0 < p \leq 2$. This new characterization of $\mathsf{h}_p^c(\mathcal{M})$ yields the following description of its dual space.

Theorem 2.3.3. *Let $0 < p \leq 2$ and q be determined by $\frac{1}{q} = 1 - \frac{1}{p}$. Then the dual space of $\mathsf{h}_p^c(\mathcal{M})$ coincide with the L_2-martingales x for which $M_q^c(x) = \sup_W \Big[\tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big)\Big]^{1/2} < \infty$. More precisely,*

(i) *Every L_2-martingale x such that $M_q^c(x) < \infty$ defines a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ by*
$$\phi_x(y) = \tau(yx^*) \text{ for } y \in L_2(\mathcal{M}).$$

(ii) *Conversely, any continuous linear functional ϕ on $\mathsf{h}_p^c(\mathcal{M})$ is given as above by some x such that $M_q^c(x) < \infty$.*

Similarly, the dual space of $\mathsf{h}_p^r(\mathcal{M})$ coincide with the L_2-martingales x for which $M_q^r(x) = M_q^c(x^*) < \infty$.

Proof. Let x be such that $M_q^c(x) < \infty$. Then x defines a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ by $\phi_x(y) = \tau(yx^*)$ for $y \in L_2(\mathcal{M})$. To see this fix $\{w_n\} \in W$. The Cauchy-Schwarz inequality gives

$$\begin{aligned}
\tau(yx^*) &= \sum_{n\geq 0} \tau\Big((dy_{n+1}w_n^{1/2-1/p})(dx_{n+1}w_n^{1/2-1/q})^*\Big) \\
&\leq \Big(\sum_{n\geq 0} \tau(w_n^{1-2/p}|dy_{n+1}|^2)\Big)^{1/2}\Big(\sum_{n\geq 0} \tau(w_n^{1-2/q}|dx_{n+1}|^2)\Big)^{1/2} \\
&\leq \Big(\sum_{n\geq 0} \tau(w_n^{1-2/p}|dy_{n+1}|^2)\Big)^{1/2} M_q^c(x).
\end{aligned}$$

Taking the infimum over W we obtain $\tau(yx^*) \leq N_p^c(y) M_q^c(x)$.

Conversely, let ϕ be a continuous linear functional on $\mathsf{h}_p^c(\mathcal{M})$ of norm ≤ 1. As $L_2(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M})$, ϕ induces a continuous linear functional on $L_2(\mathcal{M})$. Thus there exists $x \in L_2(\mathcal{M})$ such that $\phi(y) = \tau(yx^*)$ for $y \in L_2(\mathcal{M})$. By the density of $L_2(\mathcal{M})$ in $\mathsf{h}_p^c(\mathcal{M})$ we have

$$\|\phi\|_{(\mathsf{h}_p^c)^*} = \sup_{y\in L_2(\mathcal{M}), \|y\|_{\mathsf{h}_p^c}\leq 1} |\tau(yx^*)| \leq 1.$$

Thus by Proposition 2.3.2 we obtain

$$\sup_{y\in L_2(\mathcal{M}), N_p^c(y)\leq 1} |\tau(yx^*)| \leq 1. \tag{2.3.3}$$

We want to show that $M_q^c(x) < \infty$. Fix $\{w_n\} \in W$. Let y be the martingale defined by $dy_{n+1} = dx_{n+1}w_n^{1-2/q}, \forall n \in \mathbb{N}$. By (2.3.3) we have

$$\begin{aligned}
\tau(yx^*) &= \tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big) \leq N_p^c(y) \\
&\leq \tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big)^{1/2}.
\end{aligned}$$

Thus

$$\tau\Big(\sum_{n\geq 0} w_n^{1-2/q}|dx_{n+1}|^2\Big) \leq 1, \quad \forall\{w_n\} \in W.$$

Taking the supremum over W we obtain $M_q^c(x) \leq 1$.

Passing to adjoints yields the description of the continuous linear functionals on $\mathsf{h}_p^r(\mathcal{M})$.

$\square$

Remark that for $-\infty < 1/q \leq 1/2$, M_q^c and M_q^r define two norms. Let X_q^c (resp. X_q^r) be the Banach space consisting of the L_2-martingales x for which $M_q^c(x)$ (resp. $M_q^r(x)$) is finite. Theorem 2.3.3 shows that $(\mathsf{h}_p^c(\mathcal{M}))^* = X_q^c$ and $(\mathsf{h}_p^r(\mathcal{M}))^* = X_q^r$ for $0 < p \leq 2$, $\frac{1}{q} = 1 - \frac{1}{p}$.

For $-\infty < 1/q \leq 1/2$, note that $M_q^c(x)$ can be rewritten in the following form. Given $\{w_n\}_{n\geq 0} \in W$ we put

$$g_n = (w_n^{2/s} - w_{n-1}^{2/s})^{1/2}, \quad \forall n \geq 1$$

where $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It is clear that

$$\{g_n\}_{n\geq 1} \in G = \Big\{\{h_n\}_{n\geq 1}; h_n \in L_s(\mathcal{M}_n), \tau\Big(\Big(\sum_{n\geq 1} |h_n|^2\Big)^{s/2}\Big) \leq 1\Big\}.$$

Then

$$M_q^c(x) = \sup_G \Big[\tau\Big(\sum_{n\geq 1} |g_n|^2 \mathcal{E}_n |x - x_n|^2\Big)\Big]^{1/2}.$$

It is now easy to see that the dual form of Junge's noncommutative Doob maximal inequality ([20]) implies that for $q \geq 2, X_q^c = L_q^c\mathsf{mo}(\mathcal{M})$ with equivalent norms, where $L_q^c\mathsf{mo}(\mathcal{M})$ is defined in Chapter 1.

Similarly, we have $X_q^r = L_q^r\mathsf{mo}(\mathcal{M})$ with equivalent norms.

Thus for $1 \leq p \leq 2$, Theorem 2.3.3 gives another proof of the duality obtained in Chapter 1 between $\mathsf{h}_p(\mathcal{M})$ and $L_q\mathsf{mo}(\mathcal{M})$ for $\frac{1}{p} + \frac{1}{q} = 1$. Note that this new proof is much simpler and yields a better constant for the upper estimate, that is $\sqrt{p/2}$ instead of $\sqrt{2}$.

For $0 < p < 1$, Theorem 2.3.3 leads to a first description of the dual space of $\mathsf{h}_p(\mathcal{M})$. However, this description is not satisfactory. Following the classical case, we would like to describe this dual space as the Lipschitz space $\Lambda_\alpha^c(\mathcal{M})$ defined in the previous section as the dual space of $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$. Thus the description of the dual space of $\mathsf{h}_p(\mathcal{M})$ for $0 < p < 1$ is closely related to the atomic decomposition of $\mathsf{h}_p(\mathcal{M})$.

2.4 Interpolation of h_p spaces

It is a rather easy matter to identify interpolation spaces between commutative or noncommutative L_p-spaces by real or complex method. However, we need more efforts to establish interpolation results between Hardy spaces of martingales (see [19], and also [52]). Musat ([31]) extended Janson and Jones' interpolation theorem for Hardy spaces of martingales to the noncommutative setting. She proved in particular that for $1 \leq q < q_\theta < \infty$

$$(\mathcal{BMO}^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))_{\frac{q}{q_\theta}} = \mathcal{H}_{q_\theta}^c(\mathcal{M}). \tag{2.4.1}$$

See also [22] for a different proof with better constants. This section is devoted to showing the analogue of (2.4.1) in the conditioned case. Our approach is simpler and more elementary than Musat's and also valid for her situation.

We refer to [2] for details on interpolation. Recall that the noncommutative L_p-spaces associated with a semifinite von Neumann algebra form interpolation scales with respect to the complex method and the real method. More precisely, for $0<\theta<1, 1\le p_0<p_1\le\infty$ and $1\le q_0,q_1,q\le\infty$ we have

$$L_p(\mathcal{M}) = (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta \quad \text{(with equal norms)} \tag{2.4.2}$$

and

$$L_{p,q}(\mathcal{M}) = (L_{p_0,q_0}(\mathcal{M}), L_{p_1,q_1}(\mathcal{M}))_{\theta,q} \quad \text{(with equivalent norms)} \tag{2.4.3}$$

where $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, and where $L_{p,q}(\mathcal{M})$ denotes the noncommutative Lorentz space on $(\mathcal{M},\tau)$.

We can now state the main result of this section which deals with complex interpolation between the column spaces $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{h}_1^c(\mathcal{M})$.

Theorem 2.4.1. *Let $1<p<\infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p^c(\mathcal{M}). \tag{2.4.4}$$

Remark 2.4.2. All spaces considered here are compatible in the sense that they can be embedded in the $*$-algebra of measurable operators with respect to $(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathbb{N}^2)), \tau\otimes\mathrm{Tr})$. Indeed, for each $1\le p<\infty$, $\mathsf{h}_p^c(\mathcal{M})$ can be identified with a subspace of $L_p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathbb{N}^2)))$. Recall that $\mathsf{h}_p^c(\mathcal{M})$ is also defined as the closure in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ of all finite martingale differences in $\mathcal{M}$. Here $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ is the subspace of $L_p(\mathcal{M},\ell_2^c(\mathbb{N}^2))$ introduced by Junge [20] consisting of all double indexed sequences (x_{nk}) such that $x_{nk}\in L_p(\mathcal{M}_n)$ for all $k\in\mathbb{N}$. We refer to [35] for details on the column and row spaces $L_p(\mathcal{M},\ell_2^c)$ and $L_p(\mathcal{M},\ell_2^r)$. Furthermore, by the Hölder inequality and duality, recalling that the trace is finite, we have, for $1\le p<q<\infty$, the continuous inclusions

$$L_\infty(\mathcal{M}) \subset \mathsf{bmo}^c(\mathcal{M}) \subset \mathsf{h}_q^c(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M}).$$

The first inclusion is proved by (2.2.1). The second one comes from the third one by duality. Indeed, it is proved in [24] that for $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, we have $(\mathsf{h}_p^c(\mathcal{M}))^*=\mathsf{h}_{p'}^c(\mathcal{M})$, and, as already mentioned above, we have $(\mathsf{h}_1^c(\mathcal{M}))^*=\mathsf{bmo}^c(\mathcal{M})$ (see Chapter 1). Note that $L_\infty(\mathcal{M})$ is dense in all spaces above, except $\mathsf{bmo}^c(\mathcal{M})$. This implies that $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{h}_q^c(\mathcal{M})$ are dense in $\mathsf{h}_p^c(\mathcal{M})$ for $1\le p<q<\infty$.

We will need Wolff's interpolation theorem (see [51]). This result states that given Banach spaces E_i $(i=1,2,3,4)$ such that $E_1\cap E_4$ is dense in both E_2 and E_3, and

$$E_2=(E_1,E_3)_\theta \quad \text{and} \quad E_3=(E_2,E_4)_\phi$$

for some $0<\theta,\phi<1$, then

$$E_2=(E_1,E_4)_\varsigma \quad \text{and} \quad E_3=(E_1,E_4)_\xi, \tag{2.4.5}$$

where $\varsigma=\frac{\theta\phi}{1-\theta+\theta\phi}$ and $\xi=\frac{\phi}{1-\theta+\theta\phi}$. The main step of the proof of Theorem 2.4.1 is the following lemma which is based on the equivalent quasinorm N_p^c of $\|\cdot\|_{\mathsf{h}_p^c}$ described in the previous section.

Lemma 2.4.3. *Let $1<p<\infty$ and $0<\theta<1$. Then, the following holds with equivalent norms*

$$(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta = \mathsf{h}_q^c(\mathcal{M}), \tag{2.4.6}$$

where $\frac{1-\theta}{1}+\frac{\theta}{p}=\frac{1}{q}$.

Proof. **Step 1:** We first prove (2.4.6) in the case $1 < q < p \leq 2$. As explained in Remark 2.4.2, $\mathsf{h}_p^c(\mathcal{M})$ can be identified with a subspace of $L_p(\mathcal{M}\overline{\otimes}B(\ell_2(\mathbb{N}^2)))$. Thus the interpolation between noncommutative L_p-spaces in (2.4.2) gives the inclusion $(\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta \subset \mathsf{h}_q^c(\mathcal{M})$.

The reverse inclusion needs more efforts. This can be shown using the equivalent quasinorm N_p^c of $\|\cdot\|_{\mathsf{h}_p^c}$ defined previously. Let x be an L_2-finite martingale such that $\|x\|_{\mathsf{h}_q^c} < 1$. By (2.3.2) we have

$$N_q^c(x) = \inf_W \Big[\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big)\Big]^{1/2} < \Big(\frac{2}{q}\Big)^{1/2}.$$

Let $\{w_n\} \in W$ be such that

$$\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big) < \frac{2}{q}. \tag{2.4.7}$$

For $\varepsilon > 0$ and $z \in S$ we define

$$\begin{aligned} f_\varepsilon(z) &= \exp(\varepsilon(z^2-\theta^2))\sum_n dx_{n+1}w_n^{\frac{1}{2}-\frac{1}{q}}w_n^{\frac{1-z}{1}+\frac{z}{p}-\frac{1}{2}} \\ &= \exp(\varepsilon(z^2-\theta^2))\sum_n dx_{n+1}w_n^{1-(1-\frac{1}{p})z-\frac{1}{q}}. \end{aligned}$$

Then f_ε is continuous on S, analytic on S_0 and $f_\varepsilon(\theta) = x$. The term $\exp(\varepsilon(z^2-\theta^2))$ ensure that $f_\varepsilon(it)$ and $f_\varepsilon(1+it)$ tend to 0 as t goes to infinity. A direct computation gives for all $t \in \mathbb{R}$

$$\tau\Big(\sum_n w_n^{-1}|d(f_\varepsilon)_{n+1}(it)|^2\Big) = \exp(-2\varepsilon(t^2+\theta^2))\tau\Big(\sum_n w_n^{1-2/q}|dx_{n+1}|^2\Big).$$

By (2.4.7) and (2.3.2) we obtain

$$\|f_\varepsilon(it)\|_{\mathsf{h}_1^c} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2}.$$

Similarly,

$$\|f_\varepsilon(1+it)\|_{\mathsf{h}_p^c} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2}.$$

Thus $x = f_\varepsilon(\theta) \in (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta$ and

$$\|x\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \leq \exp(\varepsilon)\Big(\frac{2}{q}\Big)^{1/2};$$

whence

$$\|x\|_{(\mathsf{h}_1^c(\mathcal{M}),\mathsf{h}_p^c(\mathcal{M}))_\theta} \leq \Big(\frac{2}{q}\Big)^{1/2}\|x\|_{\mathsf{h}_q^c}.$$

Step 2: To obtain the general case, we use Wolff's interpolation theorem mentioned above. Let us first recall that for $1 < v, s, q < \infty$ and $0 < \eta < 1$ such that $\frac{1}{q} = \frac{1-\eta}{v} + \frac{\eta}{s}$, we have with equivalent norms

$$(\mathsf{h}_v^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_\eta = \mathsf{h}_q^c(\mathcal{M}). \tag{2.4.8}$$

Indeed, by Lemma 6.4 of [24], $\mathsf{h}_p^c(\mathcal{M})$ is one-complemented in $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$, for $1 \leq p < \infty$. On the other hand, for $1 < p < \infty$ the space $L_p^{\mathrm{cond}}(\mathcal{M}, \ell_2^c)$ is complemented in $L_p(\mathcal{M}, \ell_2^c(\mathbb{N}^2))$ via Stein's projection (Theorem 2.13 of [20]), and the column space $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ is a one-complemented subspace of $L_p(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathbb{N}^2)))$. Thus, we conclude from (2.4.2) that, by complementation, (2.4.8) holds.

We turn to the proof of (2.4.6). Step 1 shows that (2.4.6) holds in the case $1 < p \leq 2$. Thus it remains to deal with the case $2 < p < \infty$. We divide the proof in two cases.
Case 1: $1 < q < 2 < p < \infty$. Let $q < s < 2$. Note that $1 < q < s < p$, so there exist $0 < \theta < 1$ and $0 < \phi < 1$ such that $\frac{1-\theta}{1} + \frac{\theta}{s} = \frac{1}{q}$ and $\frac{1-\phi}{q} + \frac{\phi}{p} = \frac{1}{s}$. By (2.4.8) we have

$$\mathsf{h}_s^c(\mathcal{M}) = (\mathsf{h}_q^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\phi.$$

Furthermore, recall that $1 < q < s < 2$, so Step 1 yields

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_\theta.$$

By Wolff's interpolation theorem (2.4.5), it follows that

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\varsigma,$$

where $\varsigma = \frac{\theta\phi}{1-\theta+\theta\phi}$. A simple computation shows that $\frac{1-\varsigma}{1} + \frac{\varsigma}{p} = \frac{1}{q}$.
Case 2: $2 < q < p < \infty$. By a similar argument, we easily deduce this case from the previous one and (2.4.8) using Wolff's theorem.

Note that in both cases, the density assumption of Wolff's theorem is ensured by Remark 2.4.2. □

Lemma 2.4.4. *Let $1 < q < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_{\frac{q}{p}} = \mathsf{h}_p^c(\mathcal{M}). \tag{2.4.9}$$

Proof. Applying the duality theorem 4.5.1 of [2] to (2.4.6) we obtain (2.4.9) in the case $1 < q < p < \infty$ with $\theta = \frac{q}{p}$. Here we used the description of the dual space of $\mathsf{h}_p^c(\mathcal{M})$ for $1 \leq p < \infty$ mentioned in Remark 2.4.2. □

Proof of Theorem 2.4.1. We want to extend (2.4.9) to the case $q = 1$. To this aim we again use Wolff's interpolation theorem combined with the two previous lemmas. Let $1 < q < p < \infty$. Then there exists $0 < \phi < 1$ such that $\frac{1-\phi}{1} + \frac{\phi}{p} = \frac{1}{q}$. We set $\theta = \frac{q}{p}$. Thus by Lemma 2.4.4 we have

$$\mathsf{h}_p^c(\mathcal{M}) = (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_q^c(\mathcal{M}))_\theta.$$

Moreover we deduce from Lemma 2.4.3 that

$$\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\phi.$$

So Wolff's result yields

$$\mathsf{h}_p^c(\mathcal{M}) = (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_\varsigma,$$

where $\varsigma = \frac{\theta\phi}{1-\theta+\theta\phi}$. An easy computation gives $\varsigma = \frac{1}{p}$, and this ends the proof of (2.4.4) □

The previous results concern the conditioned column Hardy space. We now consider the whole conditioned Hardy space, and get the analogue result.

Theorem 2.4.5. *Let $1 < p < \infty$. Then, the following holds with equivalent norms*

$$(\mathsf{bmo}(\mathcal{M}), \mathsf{h}_1(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p(\mathcal{M}).$$

The proof of Theorem 2.4.5 is similar to that of Theorem 2.4.1. Indeed, we need the analogue of Lemma 2.4.3 for $\mathsf{h}_p(\mathcal{M})$, and the result will follow from the same arguments. By Wolff's result, it thus remains to show that $(\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta = \mathsf{h}_q(\mathcal{M})$ for $1 < p \leq 2$, where $\frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{q}$. Recall that for $1 \leq p \leq 2$ the space $\mathsf{h}_p(\mathcal{M})$ is defined as a sum of three components

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}).$$

We will consider each component, and then will sum the interpolation results. The following lemma describe the behaviour of complex interpolation with addition.

Lemma 2.4.6. *Let* (A_0, A_1) *and* (B_0, B_1) *be two compatible couples of Banach spaces. Then for* $0 < \theta < 1$ *we have*

$$(A_0, A_1)_\theta + (B_0, B_1)_\theta \subset (A_0 + B_0, A_1 + B_1)_\theta.$$

This result comes directly from the definition of complex interpolation.

Lemma 2.4.7. *Let* $1 \leq p_0 < p_1 \leq \infty, 0 < \theta < 1$. *Then, the following holds with equivalent norms*

$$(\mathsf{h}_{p_0}^d(\mathcal{M}), \mathsf{h}_{p_1}^d(\mathcal{M}))_\theta = \mathsf{h}_p^d(\mathcal{M})$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Recall that $\mathsf{h}_p^d(\mathcal{M})$ consists of martingale difference sequences in $\ell_p(L_p(\mathcal{M}))$. So $\mathsf{h}_p^d(\mathcal{M})$ is 2-complemented in $\ell_p(L_p(\mathcal{M}))$ for $1 \leq p \leq \infty$ via the projection

$$P : \begin{cases} \ell_p(L_p(\mathcal{M})) & \longrightarrow & \mathsf{h}_p^d(\mathcal{M}) \\ (a_n)_{n\geq 1} & \longmapsto & (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n\geq 1} \end{cases}.$$

The fact that $\ell_p(L_p(\mathcal{M}))$ form an interpolation scale with respect to the complex interpolation yields the required result. $\square$

Proof of Theorem 2.4.5. The row version of Lemma 2.4.3 holds true, as well, by considering the equivalent quasinorm N_p^r of $\|\cdot\|_{\mathsf{h}_p^r}$. The diagonal version is ensured by Lemma 2.4.7. Thus Lemma 2.4.6 yields the nontrivial inclusion $\mathsf{h}_q(\mathcal{M}) \subset (\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta$ for $1 < p \leq 2$. On the other hand, by (2.1.1) we have $\mathsf{h}_p(\mathcal{M}) = L_p(\mathcal{M})$ for $1 < p < \infty$ and (2.2.1) yields by duality the inclusion $\mathsf{h}_1(\mathcal{M}) \subset L_1(\mathcal{M})$. Hence (2.4.2) gives the reverse inclusion $(\mathsf{h}_1(\mathcal{M}), \mathsf{h}_p(\mathcal{M}))_\theta \subset \mathsf{h}_q(\mathcal{M})$ for $1 < p < \infty$. That establishs the analogue of Lemma 2.4.3 for $\mathsf{h}_p(\mathcal{M})$, and Theorem 2.4.5 follows using duality and Wolff's interpolation theorem. $\square$

We now consider the real method of interpolation. We show that the main result of this section remains true for this method. For $1 < p < \infty$ and $1 \leq r \leq \infty$, similarly to the construction of the space $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ in Remark 2.4.2 we define the column and row subspaces of $L_{p,r}(\mathcal{M}\overline{\otimes}\mathrm{B}(\ell_2(\mathbb{N}^2)))$, denoted by $L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ and $L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^r)$, respectively. Let $\mathsf{h}_{p,r}^c(\mathcal{M})$ be the space of martingales x such that $dx \in L_{p,r}^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$.

Theorem 2.4.8. *Let* $1 < p < \infty$ *and* $1 \leq r \leq \infty$. *Then, the following holds with equivalent norms*

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = \mathsf{h}_{p,r}^c(\mathcal{M}). \tag{2.4.10}$$

This result is a corollary of Theorem 2.4.1.

Proof. By a discussion similar to that at the beginning of Step 2 in the proof of Lemma 2.4.3, using (2.4.3) we can show that for $1 < v, s, q < \infty$, $1 \leq r \leq \infty$ and $0 < \eta < 1$ such that $\frac{1}{q} = \frac{1-\eta}{v} + \frac{\eta}{s}$, we have with equivalent norms

$$(\mathsf{h}_v^c(\mathcal{M}), \mathsf{h}_s^c(\mathcal{M}))_{\eta,r} = \mathsf{h}_{q,r}^c(\mathcal{M}). \tag{2.4.11}$$

We deduce (2.4.10) from (2.4.4) using the reiteration theorem on real and complex interpolations. Let $1 < p < \infty$. Consider $1 < p_0 < p < p_1 < \infty$. There exists $0 < \eta < 1$ such that

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}.$$

By Theorem 4.7.2 of [2] we obtain

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = ((\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p_0}}, (\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p_1}})_{\eta,r}.$$

Then (2.4.4) yields

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = (\mathsf{h}_{p_0}^c(\mathcal{M}), \mathsf{h}_{p_1}^c(\mathcal{M}))_{\eta,r}.$$

An application of (2.4.11) gives

$$(\mathsf{bmo}^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p},r} = \mathsf{h}_{p,r}^c(\mathcal{M}).$$

This ends the proof of (2.4.10). $\square$

Remark 2.4.9. Musat's result is a corollary of Theorem 2.4.1. By Davis' decomposition proved in Chapter 1 we have $\mathcal{H}_p^c(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M})$ for $1 \leq p < 2$. So we can show the analogue of (2.4.6) for $1 < p < 2$ as follows, for $0 < \theta < 1$ and $\frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{q}$

$$\begin{array}{lll}
& \mathcal{H}_q^c(\mathcal{M}) & \\
= & \mathsf{h}_q^c(\mathcal{M}) + \mathsf{h}_q^d(\mathcal{M}) & \\
= & (\mathsf{h}_1^c(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}))_\theta + (\mathsf{h}_1^d(\mathcal{M}), \mathsf{h}_p^d(\mathcal{M}))_\theta & \text{by Lemmas 2.4.3 and 2.4.7} \\
\subset & (\mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M}), \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M}))_\theta & \text{by Lemma 2.4.6} \\
= & (\mathcal{H}_1^c(\mathcal{M}), \mathcal{H}_p^c(\mathcal{M}))_\theta. &
\end{array}$$

On the other hand, recall that for $1 \leq p < \infty$, $\mathcal{H}_p^c(\mathcal{M})$ can be identified with the space of all L_p-martingales x such that $dx \in L_p(\mathcal{M}; \ell_2^c)$. Thus we can consider $\mathcal{H}_p^c(\mathcal{M})$ as a subspace of $L_p(\mathcal{M}\overline{\otimes}B(\ell_2))$ and the reverse inclusion follows. Then the same arguments, using duality and Wolff's theorem, yield Theorem 3.1 of [31]. Alternately, we can find Musat's result by defining an equivalent quasinorm for $\|\cdot\|_{\mathcal{H}_p^c(\mathcal{M})}, 0 < p \leq 2$ similar to N_p^c, as follows

$$\widetilde{N}_p^c(x) = \inf_W \Big[\tau\Big(\sum_n w_n^{1-2/p}|dx_n|^2\Big)\Big]^{1/2} \approx \|x\|_{\mathcal{H}_p^c(\mathcal{M})}.$$

Then all the previous proofs can be adapted to obtain the analogue results for $\mathcal{H}_p^c(\mathcal{M})$.

Appendix

In Section 2 we established the existence of an atomic decomposition for $\mathsf{h}_1(\mathcal{M})$. The problem of explicitly constructing this decomposition remains open. One encounters some substantial difficulties in trying to adapt the classical atomic construction, which used stopping times, to the noncommutative setting. Note that explicit decompositions of martingales have already been constructed to establish weak-type inequalities ([40, 41]) and a noncommutative analogue of the Gundy's decomposition ([33]). In these works, Cuculescu's projections played an important role and provide a good substitute for stopping times, which are a key tool for all these decompositions in the classical case. However, these projections do not seem to be powerful enough for the noncommutative atomic decomposition and for the noncommutative Davis' decomposition (see Chapter 1).

Problem 2.4.10. *Find a constructive proof of Theorem 2.2.4 or Theorem 2.2.9.*

Problem 2.4.11. *Construct an explicit Davis' decomposition* $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^r(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M})$.

It is also interesting to discuss the case of h_p for $0 < p < 1$. We define the noncommutative analogue of $(p, 2)$-atoms as follows.

Definition 2.4.12. *Let* $0 < p \leq 1$. $a \in L_2(\mathcal{M})$ *is said to be a* $(p, 2)_c$*-atom with respect to* $(\mathcal{M}_n)_{n\geq 1}$, *if there exist* $n \geq 1$ *and a projection* $e \in \mathcal{M}_n$ *such that*

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_2 \leq \tau(e)^{1/2-1/p}$.

Replacing (ii) *by* (ii)$'$ $l(a) \leq e$, *we get the notion of a* $(p, 2)_r$*-atom.*

We define $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ and $\mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ as in Definition 2.2.3. As for $p = 1$, we have $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M})$ contractively.

On the other hand, we can describe the dual space of $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ as a Lipschitz space. For $\alpha \geq 0$, we set

$$\Lambda_\alpha^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda_\alpha^c} < \infty\}$$

with

$$\|x\|_{\Lambda_\alpha^c} = \sup_{n\geq 1} \sup_{e\in\mathcal{P}_n} \tau(e)^{-1/2-\alpha}\tau\big(e|x - x_n|^2\big)^{1/2}.$$

By a slight modification of the proof of Theorem 2.2.6 (by setting $y_e = \frac{(x-x_n)e}{\|(x-x_n)e\|_2\tau(e)^{1/p-1/2}}$) we can show that $(\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}))^* = \Lambda_\alpha^c(\mathcal{M})$ for $0 < p \leq 1$, with $\alpha = 1/p - 1$.

At the time of this writing we do not know if $\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ coincides with $\mathsf{h}_p^c(\mathcal{M})$. The problem of the atomic decomposition of $\mathsf{h}_p(\mathcal{M})$ for $0 < p < 1$ is entirely open, and is related to Problem 2.4.10.

Problem 2.4.13. *Does one have* $\mathsf{h}_p^c(\mathcal{M}) = \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ *for* $0 < p < 1$*?*

Problem 2.4.14. *Can we describe the dual space of* $\mathsf{h}_p^c(\mathcal{M})$ *as a Lipschitz space for* $0 < p < 1$ *?*

Another perspective of research concerns the interpolation results obtained in Section 4. Recall that we define $\mathsf{h}_\infty^c(\mathcal{M})$ (resp. $\mathsf{h}_\infty^r(\mathcal{M})$) as the Banach space of the $L_\infty(\mathcal{M})$-martingales x such that $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2$ (respectively $\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k^*|^2$) converge for the weak operator topology. We set $\mathsf{h}_\infty(\mathcal{M}) = \mathsf{h}_\infty^c(\mathcal{M}) \cap \mathsf{h}_\infty^r(\mathcal{M}) \cap \mathsf{h}_\infty^d(\mathcal{M})$. At the time of this writing we do not know if the interpolation result (2.4.4) remains true if we replace $\mathsf{bmo}(\mathcal{M})$ by $\mathsf{h}_\infty(\mathcal{M})$.

Problem 2.4.15. *Does one have* $(\mathsf{h}_\infty^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_{\frac{1}{p}} = \mathsf{h}_p^c(\mathcal{M})$ *for* $1 < p < \infty$ *?*

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Chapter 3

Theory of $\mathcal{H}_p$-spaces for continuous filtrations in von Neumann algebras

Introduction

The theory of stochastic integrals and martingales with continuous time is a well-known theory with many applications. Quantum stochastic calculus is also well developed with applications reaching into fields such as quantum optics. In the setting of von Neumann algebras, many classical martingale inequalities have been reformulated for noncommutative martingales with respect to discrete filtrations, see e.g. [35, 24, 20, 27]. The aim of this paper is to study martingales with respect to continuous filtrations in von Neumann algebras. Our long term goal is to develop a satisfactory theory for semimartingales, including the convergence of the stochastic integrals. In the noncommutative setting, we cannot construct the stochastic integrals pathwise as in [8]. It is unimaginable to consider the path of a process of operators in a von Neumann algebra. However, it is well-known that in the classical case, the convergence of the stochastic integrals is closely related to the existence of the quadratic variation bracket $[\cdot,\cdot]$ via the formula

$$X_tY_t = \int^t X_{s^-}dY_s + \int^t Y_{s^-}dX_s + [X,Y]_t.$$

Here the quadratic variation bracket can be characterized as the limit in probability of the following dyadic square functions

$$[X,Y]_t = X_0Y_0 + \lim_{n\to\infty} \sum_{0\leq k<2^n} (X_{t\frac{k+1}{2^n}} - X_{t\frac{k}{2^n}})(Y_{t\frac{k+1}{2^n}} - Y_{t\frac{k}{2^n}}).$$

Hence we will first study this quadratic variation bracket in the setting of von Neumann algebras, and then deal with stochastic integrals in a forthcoming paper based on the theory developed here. More precisely, we will focus on the $L_{p/2}$-norm of this bracket by considering the Hardy spaces H_p defined in the classical case by the norm

$$\|x\|_{H_p} = \|[x,x]\|_{p/2}^{1/2}.$$

This paper develops a theory of the Hardy spaces of noncommutative martingales with respect to a continuous filtration. One fundamental application is an interpolation theory for these noncommutative function spaces which has already found applications in theory of semigroups (see e.g. [21]).

Let us consider a von Neumann algebra $\mathcal{M}$. For simplicity, we assume that $\mathcal{M}$ is finite and equipped with a normal faithful normalized trace τ. Fortunately, the theory of noncommutative H_p-spaces is now very well understood in the discrete setting, i.e., when dealing with an increasing sequence $(\mathcal{M}_n)_{n\geq 0}$ of von Neumann subalgebras of $\mathcal{M}$, whose union is weak*-dense in $\mathcal{M}$. We consider the associated conditional expectations $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$. In the noncommutative setting it is well-known that we always encounter two different objects, the row and column versions of the Hardy spaces:

$$\|x\|_{H_p^c} = \Big\| \Big(\sum_n |d_n(x)|^2 \Big)^{1/2} \Big\|_p \quad \text{and} \quad \|x\|_{H_p^r} = \Big\| \Big(\sum_n |d_n(x^*)|^2 \Big)^{1/2} \Big\|_p,$$

where $d_n(x) = \mathcal{E}_n(x) - \mathcal{E}_{n-1}(x)$. Here $\|x\|_p = (\tau(|x|^p))^{1/p}$ refers to the norm in the noncommutative L_p-space. The noncommutative Burkholder-Gundy inequalities from [35] say that

$$L_p(\mathcal{M}) = H_p \quad \text{with equivalent norms for } \ 1 < p < \infty, \tag{3.0.1}$$

where the H_p-space is defined by

$$H_p = \begin{cases} H_p^c + H_p^r & \text{for} \quad 1 \leq p < 2 \\ H_p^c \cap H_p^r & \text{for} \quad 2 \leq p < \infty \end{cases}.$$

Following the commutative theory, we should expect to define the bracket $[x, x]$ for a martingale x and then define

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x, x]\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{\widehat{\mathcal{H}}_p^r} = \|[x^*, x^*]\|_{p/2}^{1/2}.$$

Armed with the definition we may then attempt to prove (3.0.1) for a continuous filtration $(\mathcal{M}_t)_{t\geq 0}$. For simplicity, we assume that the continuous parameter set is given by the interval $[0, 1]$. We define a candidate for the noncommutative bracket following a nonstandard analysis approach. For a finite partition $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval $[0, 1]$ and $x \in \mathcal{M}$, we consider the finite bracket

$$[x, x]_\sigma = \sum_{t\in\sigma} |d_t^\sigma(x)|^2,$$

where $d_t^\sigma(x) = \mathcal{E}_t(x) - \mathcal{E}_{t^-}(x)$. Then for $p > 2$, (3.0.1) gives an a-priori bound $\|[x, x]_\sigma\|_{p/2}^{1/2} \leq \alpha_p \|x\|_p$. Hence, for a fixed ultrafilter $\mathcal{U}$ refining the general net of finite partitions of $[0, 1]$, we may simply define

$$[x, x]_{\mathcal{U}} = w\text{-}\lim_{\sigma,\mathcal{U}} [x, x]_\sigma,$$

where the weak-limit is taken in the reflexive space $L_{p/2}(\mathcal{M})$. In fact, in nonstandard analysis, the weak-limit corresponds to the standard part and is known to coincide with the classical definition of the bracket for commutative martingales. However, the norm is only lower semi-continuous with respect to the weak topology and we should not expect Burkholder/Gundy inequalities for continuous filtrations to be a simple consequence of the discrete theory of H_p-spaces. Yet, using the crucial observation that the $L_{p/2}$-norms of the discrete brackets $[x, x]_\sigma$ are monotonous up to a constant, we may show the following result.

Theorem 3.0.1. *Let $1 \leq p < \infty$ and $x \in \mathcal{M}$. Then*

$$\|[x,x]_{\mathcal{U}}\|_{p/2} \simeq \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2} \simeq \begin{cases} \sup_\sigma \|[x,x]_\sigma\|_{p/2} & \text{for} \quad 1 \leq p < 2 \\ \inf_\sigma \|[x,x]_\sigma\|_{p/2} & \text{for} \quad 2 \leq p < \infty \end{cases}.$$

In particular, this implies that the $L_{p/2}$-norm of the bracket $[x,x]_{\mathcal{U}}$ does not depend on the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm. We will discuss the independence of the bracket $[x,x]_{\mathcal{U}}$ itself from the choice of $\mathcal{U}$ in a forthcoming paper. Hence for $1 \leq p < \infty$ and $x \in \mathcal{M}$ we define the norms

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]_{\mathcal{U}}\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2}^{1/2} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

We denote by $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ respectively the corresponding completions. Theorem 3.0.1 shows that actually

$$\widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c \quad \text{with equivalent norms for} \quad 1 \leq p < \infty.$$

Hence this defines a good candidate for the Hardy space of noncommutative martingales with respect to the continuous filtration $(\mathcal{M}_t)_{0\leq t\leq 1}$. We now want to establish for this space the analogues of many well-known results in the discrete setting. For doing this, we will use the definition of the space $\mathcal{H}_p^c$, which will be more practical to work with. In particular, we may consider $\mathcal{H}_p^c$ as a subspace of some ultraproduct space, which has an L_p-module structure. Then, using ultraproduct techniques, we can show the following duality result.

Theorem 3.0.2. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$(\mathcal{H}_p^c)^* = \mathcal{H}_{p'}^c \quad \textit{with equivalent norms.}$$

Note that throughout this paper, following [35] we will consider the anti-linear duality, given by the duality bracket $(x|y) = \tau(x^*y)$. Since no confusion is possible, we will denote it by $(\mathcal{H}_p^c)^*$. With this convention, the dual space of a column space is still a column space. To prove Theorem 3.0.2, we will need to characterize the space $\mathcal{H}_p^c$ as a quotient space $\widetilde{\mathcal{H}}_p^c$ of some ultraproduct space. For $p = 1$, we also establish the analogue of the Fefferman-Stein duality in this setting:

$$(\mathcal{H}_1^c)^* = \mathcal{BMO}^c \quad \text{with equivalent norms.}$$

We have to be careful when defining the space $\mathcal{BMO}^c$. A naive candidate for the $\mathcal{BMO}^c$ norm is given by

$$\|x\|_{\mathcal{BMO}^c} = \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}, \quad \text{where} \quad \|x\|_{BMO^c(\sigma)} = \sup_{t\in\sigma} \|\mathcal{E}_t(|x - x_{t^-}|^2)\|_\infty^{1/2}.$$

However, here our restriction to finite partitions (instead of random partitions in the classical case) is restrictive. Indeed, if one of the $\|x\|_{BMO^c(\sigma)}$'s is finite, then x is already in $\mathcal{M}$. Definitively, we expect $\mathcal{BMO}^c$ to be larger than $\mathcal{M}$. We will therefore say that an element $x \in L_2(\mathcal{M})$ belongs to the unit ball of $\mathcal{BMO}^c$ if it can be approximated in L_2-norm by elements of the form

$$w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma \quad \text{in } L_2(\mathcal{M}) \quad \text{with} \quad \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{BMO^c(\sigma)} \leq 1.$$

As a consequence of Theorem 3.0.2, $\mathcal{H}_p^c$ embeds into $L_2(\mathcal{M})$ for $1 < p < 2$ and into $L_p(\mathcal{M})$ for $2 \leq p < \infty$. In fact, this still holds true for $p = 1$ by the monotonicity property. Hence we may define the Hardy space $\mathcal{H}_p$ as in the discrete setting by considering the sum of the column and row Hardy spaces in $L_2(\mathcal{M})$ for $1 \leq p < 2$, and their intersection in $L_p(\mathcal{M})$ for $2 \leq p < \infty$. The continuous analogue of (3.0.1) is then obtained by a nonstandard analysis approach, i.e., we first prove the Burkolder-Gundy inequalities at the ultraproduct level, and then take the weak-limit (i.e., the standard part).

Theorem 3.0.3. *Let $1 < p < \infty$. Then*

$$L_p(\mathcal{M}) = \mathcal{H}_p \quad \textit{with equivalent norms.}$$

We are also interested in the conditioned Hardy spaces h_p, defined in the discrete setting by the norms

$$\|x\|_{h_p^c} = \Big\| \Big(\sum_n \mathcal{E}_{n-1} |d_n(x)|^2 \Big)^{1/2} \Big\|_p, \quad \|x\|_{h_p^r} = \|x^*\|_{h_p^c} \quad \text{and} \quad \|x\|_{h_p^d} = \Big(\sum_n \|d_n(x)\|_p^p \Big)^{1/p}.$$

Then the noncommutative Burkholder inequalities proved in [24] state that

$$L_p(\mathcal{M}) = h_p \quad \text{with equivalent norms for } \ 1 < p < \infty, \tag{3.0.2}$$

where the h_p-space is defined by

$$h_p = \left\{ \begin{array}{ll} h_p^d + h_p^c + h_p^r & \text{for} \quad 1 \leq p < 2 \\ h_p^d \cap h_p^c \cap h_p^r & \text{for} \quad 2 \leq p < \infty \end{array} \right. .$$

A column version of these inequalities, which also holds true for $p = 1$, have been discovered independently in [21] and Chapter 1:

$$H_p^c = \left\{ \begin{array}{ll} h_p^d + h_p^c & \text{for} \quad 1 \leq p < 2 \\ h_p^d \cap h_p^c & \text{for} \quad 2 \leq p < \infty \end{array} \right. . \tag{3.0.3}$$

In the commutative theory the decomposition for $1 \leq p < 2$ corresponds to a version of the Davis decomposition into jump part and conditioned square function. In the conditioned case, we still have a crucial monotonicity property, and considering the conditioned bracket

$$\langle x, x \rangle_\sigma = \sum_{t \in \sigma} \mathcal{E}_{t^-} |d_t^\sigma(x)|^2$$

for a finite partition σ, we define the conditioned Hardy spaces $\widehat{\mathsf{h}}_p^c$ and h_p^c of noncommutative martingales with respect to the filtration $(\mathcal{M}_t)_{0 \leq t \leq 1}$. Then we may adapt the theory developed for the $\mathcal{H}_p^c$-spaces to $\widehat{\mathsf{h}}_p^c$ and h_p^c and obtain that

$$\widehat{\mathsf{h}}_p^c = \mathsf{h}_p^c \quad \text{with equivalent norms for } \ 1 \leq p < \infty.$$

Moreover, we can prove the conditioned analogue of Theorem 3.0.2. Note that in this case the space bmo^c is easier to describe. It is defined as the set of operators $x \in L_2(\mathcal{M})$ such that

$$\sup_{0 \leq t \leq 1} \|\mathcal{E}_t |x - x_t|^2\|_\infty < \infty.$$

To obtain the continuous analogue of the decompositions (3.0.2) and (3.0.3) for $1 < p < 2$ and $1 \leq p < 2$ respectively, we need to introduce another diagonal space $h_p^{1_c} \subset h_p^d$, which yields a stronger Davis decomposition, closer to the classical one. Then we deduce the continuous analogues of (3.0.2) and (3.0.3) for $2 \leq p < \infty$ by a dual approach. Unfortunately, we cannot directly describe the dual space of our continuous analogue of the diagonal space h_p^d. We introduce a variant of the Davis decomposition for $1 < p < 2$, based on a deep result of Randrianantoanina. This new decomposition will allow us to replace h_p^d in the sum by a larger space K_p^d. We may now describe the dual space of K_p^d, and denote it by $\mathrm{J}_{p'}^d$. Finally, defining the conditioned Hardy space by

$$\mathsf{h}_p = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c + \mathsf{h}_p^r & \text{for} \quad 1 \leq p < 2 \\ \mathrm{J}_p^d \cap \mathsf{h}_p^c \cap \mathsf{h}_p^r & \text{for} \quad 2 \leq p < \infty \end{cases},$$

we obtain the continuous analogue of (3.0.3) and (3.0.2):

Theorem 3.0.4. *Let $1 \leq p < \infty$. Then*

(i) $\mathcal{H}_p^c = \begin{cases} \mathsf{h}_p^d + \mathsf{h}_p^c & \textit{for} \quad 1 \leq p < 2 \\ \mathrm{J}_p^d \cap \mathsf{h}_p^c & \textit{for} \quad 2 \leq p < \infty \end{cases}$ *with equivalent norms.*

(ii) For $1 < p < \infty$,

$$L_p(\mathcal{M}) = \mathsf{h}_p \quad \textit{with equivalent norms.}$$

By approximation, we deduce a new characterization of $\mathcal{BMO}^c$.

At the end of the paper, we discuss the decomposition of the Hardy spaces introduced previously into algebraic atoms, and we use this decomposition to obtain the following interpolation result.

Theorem 3.0.5. *Let $1 < p < \infty$. Then*

$$\mathcal{H}_p = [\mathcal{BMO}, \mathcal{H}_1]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

The paper is organized as follows. In Section 3.1 we recall some necessary preliminaries on ultraproduct of Banach spaces in general, and on ultraproduct of von Neumann algebras in particular. We also discuss the finite case, and give some background on L_p-modules. The main part of this paper is developed in Section 3.2, where we define the Hardy space $\mathcal{H}_p$ of noncommutative martingales with respect to a continuous filtration and transfer duality results and Burkholder-Gundy inequalities from the discrete setting to this case. Section 3.3 is devoted to the study of the conditioned Hardy spaces h_p^c. It contains different characterizations of this space and some duality results. The Davis decomposition of the space $\mathcal{H}_p^c$ is presented in Section 3.4, where we introduce the diagonal spaces h_p^d and $\mathsf{h}_p^{1_c}$ for $1 \leq p < 2$. In order to pass to the duals in this decomposition, we discuss in Section 3.5 a variant way of considering the sum of two Banach spaces. In our setting this corresponds in some sense to focus on the decomposition at the level of $L_2(\mathcal{M})$, and with the help of Randrianantoanina's results we extend our continuous Davis decomposition to this stronger sum. At the end of Section 3.5 we obtain the complete Burkholder inequalities. We end this paper with a discussion on some algebraic atoms in Section 3.6, which we use immediately in Section 3.7 to establish the expected interpolation resuls. At the beginning of each section, we recall the discrete results that we want to reformulate in the continuous setting, and add some details on the discrete proofs.

Throughout this paper, the notation $a_p \simeq b_p$ means that there exist two positive constants c and C such that

$$c \leq \frac{a_p}{b_p} \leq C.$$

3.1 Preliminaries

3.1.1 Noncommutative L_p-spaces and martingales with respect to continuous filtrations

We use standard notation in operator algebras. We refer to [28, 46] for background on von Neumann algebra theory, to the survey [36] for details on noncommutative L_p-spaces, and to [13, 48] in particular for the Haagerup noncommutative L_p-spaces. In the sequel, even if we will define some L_p-spaces in the type III case, we will mainly work with noncommutative L_p-spaces associated to semifinite von Neumann algebras. Let us briefly recall this construction. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal faithful semifinite trace τ. For $0 < p \leq \infty$, we denote by $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. Note that if $p = \infty$, $L_p(\mathcal{M})$ is just $\mathcal{M}$ itself with the operator norm; also recall that for $0 < p < \infty$ the (quasi) norm on $L_p(\mathcal{M})$ is defined by

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L_p(\mathcal{M})$$

where $|x| = (x^*x)^{1/2}$ is the usual modulus of x.

Following [35], for $1 \leq p < \infty$ and a finite sequence $a = (a_n)_{n\geq 0}$ in $L_p(\mathcal{M})$ we set

$$\|a\|_{L_p(\mathcal{M};\ell_2^c)} = \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_p \quad \text{and} \quad \|a\|_{L_p(\mathcal{M};\ell_2^r)} = \|a^*\|_{L_p(\mathcal{M};\ell_2^c)}.$$

Then $\|\cdot\|_{L_p(\mathcal{M};\ell_2^c)}$ (resp. $\|\cdot\|_{L_p(\mathcal{M};\ell_2^r)}$) defines a norm on the family of finite sequences of $L_p(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L_p(\mathcal{M};\ell_2^c)$ (resp. $L_p(\mathcal{M};\ell_2^r)$). For $p = \infty$, we define $L_\infty(\mathcal{M};\ell_2^c)$ (respectively $L_\infty(\mathcal{M};\ell_2^r)$) as the Banach space of the sequences in $L_\infty(\mathcal{M})$ such that $\sum_{n\geq 0} x_n^* x_n$ (respectively $\sum_{n\geq 0} x_n x_n^*$) converges for the weak-operator topology. These spaces will be denoted by $L_p(\mathcal{M};\ell_2^c(I))$ and $L_p(\mathcal{M};\ell_2^r(I))$ when the considered sequences are indexed by I.

Let $(\mathcal{M}_t)_{t\geq 0}$ be an increasing family of von Neumann subalgebras of $\mathcal{M}$ whose union is weak*-dense in $\mathcal{M}$. Moreover, we assume that for all $t \geq 0$ there exist normal faithful conditional expectations $\mathcal{E}_t : \mathcal{M} \to \mathcal{M}_t$. Throughout this paper, we assume that the filtration $(\mathcal{M}_t)_{t\geq 0}$ is right continuous, i.e., $\mathcal{M}_t = \bigcap_{s>t} \mathcal{M}_s$ for all $t \geq 0$. A family $x = (x_t)_{t\geq 0}$ in $L_1(\mathcal{M})$ is called a noncommutative martingale with respect to $(\mathcal{M}_t)_{t\geq 0}$ if

$$\mathcal{E}_s(x_t) = x_s, \quad \forall\, 0 \leq s \leq t.$$

If in addition all x_t's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, then x is called an L_p-martingale. In this case we set

$$\|x\|_p = \sup_{t\geq 0} \|x_t\|_p.$$

If $\|x\|_p < \infty$, we say that x is a bounded L_p-martingale.

Let $x = (x_t)_{t\geq 0}$ be a noncommutative martingale with respect to $(\mathcal{M}_t)_{t\geq 0}$. We say that x is a finite martingale if there exists a finite time $T \geq 0$ such that $x_t = x_T$ for all $t \geq T$. In this paper, we will only consider finite martingales on $[0, 1]$, i.e., $T = 1$. In this case, for a finite partition $\sigma = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$ of $[0, 1]$ we denote $t^+ = t_{j+1}$ the successor of $t = t_j$ and $t^- = t_{j-1}$ its predecessor, and for $t \geq 0$ we define

$$d_t(x) = \begin{cases} x_t - x_{t^-} & \text{for } t > 0 \\ x_0 & \text{for } t = 0 \end{cases}.$$

In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_t = \mathcal{E}_t(x)$ for all $t \geq 0$.

3.1.2 Ultraproduct techniques

Ultraproduct of Banach spaces

Our approach will be mainly based on ultraproduct constructions. Let us first recall the definition and some well-known results on the ultraproducts of Banach spaces. Let $\mathcal{U}$ be an ultrafilter on a directed set $\mathcal{I}$. They are fixed throughout all this subsection. Recall that $\mathcal{U}$ is a collection of subsets of $\mathcal{I}$ such that

(i) $\emptyset \notin \mathcal{U}$;

(ii) If $A, B \subset \mathcal{I}$ such that $A \subset B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$;

(iii) If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$;

(iv) If $A \subset \mathcal{I}$, then either $A \in \mathcal{U}$ or $\mathcal{I} \setminus A \in \mathcal{U}$.

Let X be a normed vector space. For a family $(x_i)_{i\in\mathcal{I}}$ indexed by $\mathcal{I}$ in X, we say that $x = \lim_{i,\mathcal{U}} x_i$ is the limit of the x_i's along the ultrafilter $\mathcal{U}$ if

$$\{i \in \mathcal{I} \ : \ \|x - x_i\| < \varepsilon\} \in \mathcal{U} \quad \text{for all} \ \ \varepsilon > 0.$$

Recall that this limit always exists whenever the family $(x_i)_{i\in\mathcal{I}}$ is in a compact space. If X is a dual space, then its unit ball is weak*-compact, and any bounded family in X admits a weak*-limit along the ultrafilter $\mathcal{U}$. If X is reflexive, since the weak-topology coincide with the weak*-topology, we deduce that any bounded family in X admits a weak-limit along the ultrafilter $\mathcal{U}$.

We now turn to the ultraproduct construction. Let us start with the ultraproduct of a family $(X_i)_{i\in\mathcal{I}}$ of Banach spaces. Let $\ell_\infty(\{X_i : i \in \mathcal{I}\})$ be the space of bounded families $(x_i)_{i\in\mathcal{I}} \in \prod_i X_i$ equipped with the supremum norm. We define the ultraproduct $\prod_{\mathcal{U}} X_i$, also denoted by $\prod_i X_i/\mathcal{U}$, as the quotient space $\ell_\infty(\{X_i : i \in \mathcal{I}\})/\mathcal{N}^{\mathcal{U}}$, where $\mathcal{N}^{\mathcal{U}}$ denotes the (closed) subspace of $\mathcal{U}$-vanishing families, i.e.,

$$\mathcal{N}^{\mathcal{U}} = \{(x_i)_{i\in\mathcal{I}} \in \ell_\infty(\{X_i : i \in \mathcal{I}\}) \ : \ \lim_{i,\mathcal{U}} \|x_i\|_{X_i} = 0\}.$$

We will denote by $(x_i)^\bullet$ the element of $\prod_{\mathcal{U}} X_i$ represented by the family $(x_i)_{i\in\mathcal{I}}$. Recall that the quotient norm is simply given by

$$\|(x_i)^\bullet\| = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}.$$

If $X_i = X$ for all i, then we denote by $\ell_\infty(\mathcal{I}; X)$ the space of bounded X-valued families and by $\prod_{\mathcal{U}} X$ the quotient space $\ell_\infty(\mathcal{I}; X)/\mathcal{N}^{\mathcal{U}}$, called ultrapower in this case. We refer to [15, 45] for basic facts about ultraproducts of Banach spaces. If $(X_i)_{i\in\mathcal{I}}, (Y_i)_{i\in\mathcal{I}}$ are two families of Banach spaces and $T_i : X_i \to Y_i$ are linear operators uniformly bounded in $i \in \mathcal{I}$, we can define canonically the ultraproduct map $T_{\mathcal{U}} = (T_i)^\bullet$ as

$$T_{\mathcal{U}} : \left\{ \begin{array}{ccc} \prod_{\mathcal{U}} X_i & \longrightarrow & \prod_{\mathcal{U}} Y_i \\ (x_i)^\bullet & \longmapsto & (T_i x_i)^\bullet \end{array} \right. .$$

In the sequel we will often use the following useful fact without any further reference.

Lemma 3.1.1. *Let $(X_i)_{i\in\mathcal{I}}$ be a family of Banach spaces and let $x = (x_i)^\bullet \in \prod_{\mathcal{U}} X_i$ be such that $\|x\|_{\prod_{\mathcal{U}} X_i} = \lim_{i,\mathcal{U}} \|x_i\|_{X_i} < 1$. Then there exists a family $(\widetilde{x}_i)_{i\in\mathcal{I}} \in \ell_\infty(\{X_i : i \in \mathcal{I}\})$ such that*

$$x = (\widetilde{x}_i)^\bullet \quad \textit{and} \quad \|\widetilde{x}_i\|_{X_i} < 1, \ \forall\, i \in \mathcal{I}.$$

Proof. Setting

$$\widetilde{x}_i = \begin{cases} x_i & \text{if } \|x_i\|_{X_i} < 1 \\ 0 & \text{otherwise} \end{cases},$$

we get a family verifying $\|\widetilde{x}_i\|_{X_i} < 1$ for all $i \in \mathcal{I}$. Moreover, by the definition of the limit along the ultrafilter $\mathcal{U}$, we have $\lim_{i,\mathcal{U}} \|x_i - \widetilde{x}_i\|_{X_i} = 0$. Indeed, if we denote $\ell = \lim_{i,\mathcal{U}} \|x_i\|_{X_i} < 1$, then for any $\delta > 0$ we have

$$A_\delta = \{i \in \mathcal{I} \,:\, |\ell - \|x_i\|_{X_i}| < \delta\} \in \mathcal{U}.$$

Observe that for $\delta = \frac{1-\ell}{2} > 0$, each $i \in A_\delta$ satisfies $\|x_i\|_{X_i} < \ell + \delta = \frac{1+\ell}{2} < 1$. Hence for all $\varepsilon > 0$, the condition (ii) in the definition of an ultrafilter implies

$$A_{\frac{1-\ell}{2}} \subset \{i \in \mathcal{I} \,:\, \|x_i\|_{X_i} < 1\} \subset \{i \in \mathcal{I} \,:\, \|x_i - \widetilde{x}_i\|_{X_i} < \varepsilon\} \in \mathcal{U}.$$

This shows that $(x_i)^\bullet = (\widetilde{x}_i)^\bullet$ and ends the proof. □

We will need to study the dual space of an ultraproduct. For a family of Banach spaces $(X_i)_{i\in\mathcal{I}}$, there is a canonical isometric embedding J of $\prod_{\mathcal{U}} X_i^*$ into $\Big(\prod_{\mathcal{U}} X_i\Big)^*$ defined by

$$(Jx^*|x) = \lim_{i,\mathcal{U}} (x_i^*|x_i)$$

for $x^* = (x_i^*)^\bullet \in \prod_{\mathcal{U}} X_i^*$ and $x = (x_i)^\bullet \in \prod_{\mathcal{U}} X_i$. Hence we may identify $\prod_{\mathcal{U}} X_i^*$ with a subspace of $\Big(\prod_{\mathcal{U}} X_i\Big)^*$. These two spaces coincide in the following case.

Lemma 3.1.2 ([16]). *Let $(X_i)_{i\in\mathcal{I}}$ be a family of Banach spaces. Then $\Big(\prod_{\mathcal{U}} X_i\Big)^* = \prod_{\mathcal{U}} X_i^*$ if and only if $\prod_{\mathcal{U}} X_i$ is reflexive.*

Even in the non reflexive case, the subspace $\prod_{\mathcal{U}} X_i^*$ is "big" in $\Big(\prod_{\mathcal{U}} X_i\Big)^*$ in the sense of the following Lemma. This is also a well-known fact of the theory of ultraproducts (see [45], Section 11), we include a proof for the convenience of the reader.

Lemma 3.1.3. *Let $(X_i)_{i\in\mathcal{I}}$ be a family of Banach spaces. Then the unit ball of $\prod_{\mathcal{U}} X_i^*$ is weak*-dense in the unit ball of $\Big(\prod_{\mathcal{U}} X_i\Big)^*$.*

Proof. We first prove that for two normed vector spaces X and Y such that Y is a norming subspace of X^*, the unit ball of Y is weak*-dense in the unit ball of X^*. Suppose that B_Y is not weak*-dense in B_{X^*}, then by the Hahn-Banach Theorem there exist $x^* \in B_{X^*}$ and $x \in X$ such that $|(x^*|x)| = 1$ and for all $y \in B_Y$, $|(y|x)| < \delta$, $0 < \delta < 1$. Since Y is a norming subspace of X^* we have

$$\|x\|_X = \sup_{y\in B_Y} |(y|x)| < \delta.$$

Then

$$1 = |(x^*|x)| \leq \|x^*\|_{X^*}\|x\|_X < \delta,$$

which contradicts $\delta < 1$. It remains to apply this general result to $X = \prod_{\mathcal{U}} X_i$ and $Y = \prod_{\mathcal{U}} X_i^*$. It suffices to see that $\prod_{\mathcal{U}} X_i^*$ is a norming subspace of $\Big(\prod_{\mathcal{U}} X_i\Big)^*$. Let $x = (x_i)^\bullet \in \prod_{\mathcal{U}} X_i$. For each $i \in \mathcal{I}$, there exists $z_i^* \in B_{X_i^*}$ such that $\|x\|_{X_i} = |(z_i^*|x_i)|$.

Multiplying by a complex number of modulus 1, we can assume that $\|x\|_{X_i} = (z_i^*|x_i)$. Thus

$$\begin{aligned}\|x\|_{\prod_{\mathcal{U}} X_i} &= \lim_{i,\mathcal{U}} \|x_i\|_{X_i} = \lim_{i,\mathcal{U}} (z_i^*|x_i) \\ &\leq \sup_{y^*=(y_i^*)^\bullet \in B_{\prod_{\mathcal{U}} X_i^*}} |\lim_{i,\mathcal{U}} (y_i^*|x_i)| = \sup_{y^* \in B_{\prod_{\mathcal{U}} X_i^*}} |(y^*|x)|.\end{aligned}$$

□

Ultraproduct of von Neumann algebras : the general case

We now consider the ultraproduct construction for von Neumann algebras. For convenience we will simply consider ultrapowers, but all the following discussion remains valid for ultraproducts. It is well-known that if $\mathcal{A}$ is a C*-algebra, then $\prod_{\mathcal{U}} \mathcal{A}$ is still a C*-algebra. On the other hand, the class of von Neumann algebras is not closed under ultrapowers. However, according to Groh's work [11], we know that the class of the preduals of von Neumann algebras is closed under ultrapowers. Let $\mathcal{M}$ be a von Neumann algebra. Then $\prod_{\mathcal{U}} \mathcal{M}_*$ is the predual of a von Neumann algebra denoted by

$$\widetilde{\mathcal{M}}_{\mathcal{U}} = \left(\prod_{\mathcal{U}} \mathcal{M}_*\right)^*.$$

Moreover, $\prod_{\mathcal{U}} \mathcal{M}$ identifies naturally to a weak*-dense subalgebra of $\widetilde{\mathcal{M}}_{\mathcal{U}}$. As detailed in [42], we can also see $\widetilde{\mathcal{M}}_{\mathcal{U}}$ as the von Neumann algebra generated by $\prod_{\mathcal{U}} \mathcal{M}$ in $B(\prod_{\mathcal{U}} \mathcal{H})$, where we have a standard $*$-representation of $\mathcal{M}$ over the Hilbert space $\mathcal{H}$. Following Raynaud's work [42], for all $p > 0$ we can construct an isometric isomorphism

$$\Lambda_p : \prod_{\mathcal{U}} L_p(\mathcal{M}) \to L_p(\widetilde{\mathcal{M}}_{\mathcal{U}}),$$

which preserves the following structures

- conjugation: $\Lambda_p((x_i^*)^\bullet) = \Lambda_p((x_i)^\bullet)^*$,
- absolute values: $\Lambda_p((|x_i|)^\bullet) = |\Lambda_p((x_i)^\bullet)|$,
- $\prod_{\mathcal{U}} \mathcal{M}$-bimodule structure: $\Lambda_p((a_i)^\bullet \cdot (x_i)^\bullet \cdot (b_i)^\bullet) = (a_i)^\bullet \cdot \Lambda_p((x_i)^\bullet) \cdot (b_i)^\bullet$,
- external product: $\Lambda_r((x_i)^\bullet \cdot (y_i)^\bullet) = \Lambda_p((x_i)^\bullet) \cdot \Lambda_q((y_i)^\bullet)$ for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

for all $(x_i)^\bullet \in \prod_{\mathcal{U}} L_p(\mathcal{M})$, $(y_i)^\bullet \in \prod_{\mathcal{U}} L_q(\mathcal{M})$ and $(a_i)^\bullet, (b_i)^\bullet \in \prod_{\mathcal{U}} \mathcal{M}$. In the sequel we will identify the spaces $\prod_{\mathcal{U}} L_p(\mathcal{M})$ and $L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})$ without any further reference.

Ultraproduct of von Neumann algebras : the finite case

We now discuss the finite situation. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful normalized trace τ. In this case the usual von Neumann algebra ultrapower is $\mathcal{M}_{\mathcal{U}} = \ell_\infty(\mathcal{I}; X)/\mathcal{I}^{\mathcal{U}}$, where

$$\mathcal{I}^{\mathcal{U}} = \{(x_i)_{i\in\mathcal{I}} \in \ell_\infty(\mathcal{I}; X) \ : \ \lim_{i,\mathcal{U}} \tau(x_i^* x_i) = 0\}.$$

According to Sakai ([44]), $\mathcal{M}_{\mathcal{U}}$ is a finite von Neumann algebra when equipped with the ultrapower map of the trace τ, denoted by $\tau_{\mathcal{U}}$ and defined by

$$\tau_{\mathcal{U}}((x_i)^\bullet) = \lim_{i,\mathcal{U}} \tau(x_i).$$

Note that this definition is compatible with $\mathcal{I}_{\mathcal{U}}$, and defines a normal faithful normalized trace on $\mathcal{M}_{\mathcal{U}}$. We may identify $\mathcal{M}_{\mathcal{U}}$ as a dense subspace of $L_1(\mathcal{M}_{\mathcal{U}})$ via the map $x \in \mathcal{M}_{\mathcal{U}} \mapsto \tau_{\mathcal{U}}(x\ \cdot\ ) \in L_1(\mathcal{M}_{\mathcal{U}})$. Then for $x = (x_i)^\bullet \in \mathcal{M}_{\mathcal{U}}$, we have $\|x\|_1 = \lim_{i,\mathcal{U}} \|x_i\|_1$. Observe that this does not depend on the representing family (x_i) of x. Let us define the map

$$\iota : \left\{ \begin{array}{ccc} \mathcal{M}_{\mathcal{U}} & \longrightarrow & L_1(\widetilde{\mathcal{M}}_{\mathcal{U}}) \\ (x_i)^\bullet & \longmapsto & (\tau(x_i\ \cdot\ ))^\bullet \end{array} \right. .$$

We see that this map is well-defined, and it is clear that $\|\iota((x_i)^\bullet)\|_1 = \lim_{i,\mathcal{U}} \|x_i\|_1$. Hence by density we can extend ι to an isometry from $L_1(\mathcal{M}_{\mathcal{U}})$ into $L_1(\widetilde{\mathcal{M}}_{\mathcal{U}})$. Since $L_1(\mathcal{M}_{\mathcal{U}})$ is stable under $\widetilde{\mathcal{M}}_{\mathcal{U}}$ actions, Theorem III.2.7 of [46] gives a central projection $e_{\mathcal{U}}$ in $\widetilde{\mathcal{M}}_{\mathcal{U}}$ such that $L_1(\mathcal{M}_{\mathcal{U}}) = L_1(\widetilde{\mathcal{M}}_{\mathcal{U}})e_{\mathcal{U}}$. We can see that $e_{\mathcal{U}}$ is the support projection of the trace $\tau_{\mathcal{U}}$. In the sequel we will identify $\mathcal{M}_{\mathcal{U}}$ as a subalgebra of $\widetilde{\mathcal{M}}_{\mathcal{U}}$, by considering $\mathcal{M}_{\mathcal{U}} = \widetilde{\mathcal{M}}_{\mathcal{U}}e_{\mathcal{U}}$. More generally we have

$$L_p(\mathcal{M}_{\mathcal{U}}) = L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})e_{\mathcal{U}} \quad \text{for all } 0 < p \leq \infty. \tag{3.1.1}$$

The subspace $L_p(\mathcal{M}_{\mathcal{U}})$ can be characterized by using the notion of p-equiintegrability as follows. Let us recall the definition of a p-equiintegrable subset of a noncommutative L_p-space introduced in [46] for $p = 1$ and by Randrianantoanina in [38] for any p.

Definition 3.1.4. *Let $0 < p < \infty$. A bounded subset K of $L_p(\mathcal{M})$ is called p-equiintegrable if*

$$\lim_{n\to\infty} \sup_{x\in K} \|e_n x e_n\|_p = 0$$

for every decreasing sequences $(e_n)_n$ of projections of $\mathcal{M}$ which weak-converges to 0.*
If $p = 1$, we say that K is uniformly integrable.

Recall that finite subsets of $L_p(\mathcal{M})$ are p-equiintegrable. We will use the following characterization coming from Corollary 2.7 of [14].

Lemma 3.1.5. *Let $1 \leq p < \infty$ and $(x_i)_{i\in\mathcal{I}}$ be a bounded family in $L_p(\mathcal{M})$. Then the following assertions are equivalent.*

(i) $(x_i)_{i\in\mathcal{I}}$ *is p-equiintegrable;*

(ii) $\lim_{T\to\infty} \sup_i \operatorname{dist}_{L_p}(x_i, TB_{\mathcal{M}}) = 0$;

(iii) $\lim_{T\to\infty} \lim_{i,\mathcal{U}} \|x_i \mathbb{1}(|x_i| > T)\|_p = 0$,

where for $a \geq 0$, $\mathbb{1}(a > T)$ denotes the spectral projection of a corresponding to the interval (T, ∞).

Observe that (3.1.1) implies that for $0 < p < \infty$ and $x \in L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})$

$$x \in L_p(\mathcal{M}_{\mathcal{U}}) \Leftrightarrow x = xe_{\mathcal{U}}.$$

Moreover, in the finite case, $e_{\mathcal{U}}$ corresponds to the projection denoted by s_e in [43]. Hence Theorem 4.6 of [43] yields the following characterization of $L_p(\mathcal{M}_{\mathcal{U}})$.

Theorem 3.1.6. *Let $0 < p < \infty$ and $x \in L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})$. Then the following assertions are equivalent.*

(i) $x \in L_p(\mathcal{M}_{\mathcal{U}})$;

(ii) x admits a p-equiintegrable representing family $(x_i)_{i\in\mathcal{I}}$.

For $0 < p < \widetilde{p} \leq \infty$, since $\mathcal{M}$ is finite we have a contractive inclusion $L_{\widetilde{p}}(\mathcal{M}) \subset L_p(\mathcal{M})$. Let us denote by $I_{\widetilde{p},p} : \prod_{\mathcal{U}} L_{\widetilde{p}}(\mathcal{M}) \to \prod_{\mathcal{U}} L_p(\mathcal{M})$ the contractive ultraproduct map of the componentwise inclusion maps. Note that although the componentwise inclusion maps are injective, the ultraproduct map $I_{\widetilde{p},p}$ is not. However, its restriction to $L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}})$ is injective. Indeed, using the weak*-density of $\prod_{\mathcal{U}} \mathcal{M}$ in $\widetilde{\mathcal{M}_{\mathcal{U}}}$, we see that $I_{\widetilde{p},p}$ is bimodular under the action of $\widetilde{\mathcal{M}_{\mathcal{U}}}$. Hence, if $x \in L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}})$ satisfies $x = xe_{\mathcal{U}}$, then $I_{\widetilde{p},p}(x) = I_{\widetilde{p},p}(xe_{\mathcal{U}}) = I_{\widetilde{p},p}(x)e_{\mathcal{U}} \in L_p(\mathcal{M}_{\mathcal{U}})$. This shows that $I_{\widetilde{p},p} : L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}}) \to L_p(\mathcal{M}_{\mathcal{U}})$. Moreover, since $\mathcal{M}_{\mathcal{U}}$ is finite, the map $I_{\widetilde{p},p}$ coincides on $L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}})$ with the natural inclusion $L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}}) \subset L_p(\mathcal{M}_{\mathcal{U}})$. We deduce from Theorem 3.1.6 the following description of the space $L_p(\mathcal{M}_{\mathcal{U}})$, viewed as a subspace of $L_p(\widetilde{\mathcal{M}_{\mathcal{U}}})$.

Lemma 3.1.7. *Let $0 < p < \infty$. Then*

$$L_p(\mathcal{M}_{\mathcal{U}}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}}))}^{\|\cdot\|_{L_p(\widetilde{\mathcal{M}_{\mathcal{U}}})}}.$$

Proof. Let us first show that $I_{\widetilde{p},p}(L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}})) \subset L_p(\mathcal{M}_{\mathcal{U}})$ for $\widetilde{p} > p$. Let $x = (x_i)^\bullet \in L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}})$. By Theorem 3.1.6, it suffices to prove that the family $(x_i)_{i\in\mathcal{I}}$ is p-equiintegrable. For $T > 0$ and each $i \in \mathcal{I}$ we have

$$\|x_i \mathbb{1}(|x_i| > T)\|_p \leq \|x_i |x_i|^{\frac{\widetilde{p}}{p}-1} T^{1-\frac{\widetilde{p}}{p}}\|_p \leq T^{1-\frac{\widetilde{p}}{p}} \|x_i\|_{\widetilde{p}}^{\frac{\widetilde{p}}{p}}.$$

Taking the limit along the ultrafilter $\mathcal{U}$ we obtain

$$\lim_{i,\mathcal{U}} \|x_i \mathbb{1}(|x_i| > T)\|_p \leq T^{1-\frac{\widetilde{p}}{p}} \|x\|_{L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}})}^{\frac{\widetilde{p}}{p}}.$$

Since $1-\frac{\widetilde{p}}{p} < 0$, this tends to 0 as T goes to ∞. We conclude that $(x_i)_{i\in\mathcal{I}}$ is p-equiintegrable by using Lemma 3.1.5. Conversely, let $x \in L_p(\mathcal{M}_{\mathcal{U}})$. Since $\mathcal{M}_{\mathcal{U}}$ is finite, $L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}})$ is dense in $L_p(\mathcal{M}_{\mathcal{U}})$ for all $\widetilde{p} > p$. Hence for all $\varepsilon > 0$ there exists $y \in L_{\widetilde{p}}(\mathcal{M}_{\mathcal{U}})$ such that $\|x - y\|_{L_p(\mathcal{M}_{\mathcal{U}})} < \varepsilon$. Since $L_p(\mathcal{M}_{\mathcal{U}})$ is isometrically embedded into $L_p(\widetilde{\mathcal{M}_{\mathcal{U}}})$ and $y = I_{\widetilde{p},p}(y) \in I_{\widetilde{p},p}(L_{\widetilde{p}}(\widetilde{\mathcal{M}_{\mathcal{U}}}))$, this ends the proof. □

For $p = 1$, we can translate the notion of uniform integrability in terms of compactness as follows.

Theorem 3.1.8 ([46])**.** *Let K be a bounded subset of the predual $\mathcal{M}_*$ of $\mathcal{M}$. Then the following assertions are equivalent.*

(i) K is uniformly integrable;

(ii) K is weakly relatively compact.

Let us consider

$$i_{\mathcal{U}} : \left\{ \begin{array}{ccc} (\mathcal{M}, \tau) & \longrightarrow & (\mathcal{M}_{\mathcal{U}}, \tau_{\mathcal{U}}) \\ x & \longmapsto & (x)^\bullet \end{array} \right. .$$

Since $i_{\mathcal{U}}$ is trace preserving, this yields an isometric embedding of $L_1(\mathcal{M})$ into $L_1(\mathcal{M}_{\mathcal{U}})$. Hence we get natural inclusions

$$L_1(\mathcal{M}) \subset L_1(\mathcal{M}_{\mathcal{U}}) \subset L_1(\widetilde{\mathcal{M}_{\mathcal{U}}}),$$

where $L_1(\widetilde{\mathcal{M}_{\mathcal{U}}})$ represents the bounded families in $L_1(\mathcal{M})$, $L_1(\mathcal{M}_{\mathcal{U}})$ corresponds to the weakly converging families along $\mathcal{U}$ and $L_1(\mathcal{M})$ consists of the collection of the constants families.

We end this subsection with the introduction of a conditional expectation. We set

$$\mathcal{E}_{\mathcal{U}} = (i_{\mathcal{U}})^* : \mathcal{M}_{\mathcal{U}} \to \mathcal{M}.$$

Then $\mathcal{E}_{\mathcal{U}}$ is a normal faithful conditional expectation on $\mathcal{M}_{\mathcal{U}}$. Since $\mathcal{E}_{\mathcal{U}}$ is trace preserving, for all $1 \leq p \leq \infty$ we can extend $\mathcal{E}_{\mathcal{U}}$ to a contraction from $L_p(\mathcal{M}_{\mathcal{U}})$ onto $L_p(\mathcal{M})$, still denoted by $\mathcal{E}_{\mathcal{U}}$. Moreover, for $1 < p \leq \infty$ and $x = (x_i)^\bullet \in L_p(\mathcal{M}_{\mathcal{U}})$ we have

$$\mathcal{E}_{\mathcal{U}}(x) = w^*\text{-}\lim_{i,\mathcal{U}} x_i,$$

where the weak*-limit is taken in $L_p(\mathcal{M})$. Indeed, for $y \in L_{p'}(\mathcal{M})$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we can write

$$\tau(\mathcal{E}_{\mathcal{U}}(x)^* y) = \tau_{\mathcal{U}}(x^* i_{\mathcal{U}}(y)) = \lim_{i,\mathcal{U}} \tau(x_i^* y). \tag{3.1.2}$$

Note that since in this case $L_p(\mathcal{M})$ is a dual space, the weak*-limit of the x_i's exists for any bounded family (x_i). Hence we may extend $\mathcal{E}_{\mathcal{U}}$ to $L_p(\widetilde{\mathcal{M}_{\mathcal{U}}})$ for $1 < p \leq \infty$. However this extension, still denoted by $\mathcal{E}_{\mathcal{U}}$ in the sequel, is no longer faithful. For $1 < p < \infty$, since $L_p(\mathcal{M})$ is reflexive, the weak*-limit corresponds to the weak-limit. Recall that by Theorem 3.1.8, $L_1(\mathcal{M}_{\mathcal{U}})$ corresponds to the weakly converging families. Thus (3.1.2) implies that for $1 \leq p < \infty$ and $x = (x_i)^\bullet \in L_p(\mathcal{M}_{\mathcal{U}})$ we have

$$\mathcal{E}_{\mathcal{U}}(x) = w\text{-}\lim_{i,\mathcal{U}} x_i.$$

3.1.3 L_p $\mathcal{M}$-modules

We will use the theory of L_p-modules introduced in [23]. This structure will help us to prove duality and interpolation results for different $\mathcal{H}_p$-spaces. We may say that L_p-modules are L_p-versions of Hilbert W^*-modules. Let $\mathcal{M}$ be a von Neumann algebra.

Definition 3.1.9. *Let $1 \leq p < \infty$. A right $\mathcal{M}$-module X is called a right L_p $\mathcal{M}$-module if it has an $L_{p/2}(\mathcal{M})$-valued inner product, i.e. there is a sesquilinear map $\langle \cdot, \cdot \rangle : X \times X \to L_{p/2}(\mathcal{M})$, conjugate linear in the first variable, such that for all $x, y \in X$ and all $a \in \mathcal{M}$*

(i) $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,

(ii) $\langle x, y \rangle^* = \langle y, x \rangle$,

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$,

and X is complete in the inherited (quasi)norm

$$\|x\| = \|\langle x, x \rangle\|_{p/2}^{1/2}.$$

We call X a right L_∞ $\mathcal{M}$-module if it has an $L_\infty(\mathcal{M})$-valued inner product and is complete with respect to the strong operator topology, i.e. the topology arising from the seminorms

$$\|x\|_\varphi = (\varphi(\langle x, x \rangle))^{1/2}, \quad \varphi \in \mathcal{M}_*^+.$$

The basic example of such a right L_p $\mathcal{M}$-module is given by the column L_p-space $L_p(\mathcal{M};\ell_2^c)$. Here for $a\in\mathcal{M}$ and $x=\sum_{n\geq 0} e_{n,0}\otimes x_n, y=\sum_{n\geq 0} e_{n,0}\otimes y_n\in L_p(\mathcal{M};\ell_2^c)$ we define the right $\mathcal{M}$-module action by

$$x\cdot a=\sum_{n\geq 0} e_{n,0}\otimes (x_n a).$$

Then we define the following $L_{p/2}(\mathcal{M})$-valued inner product

$$\langle x,y\rangle_{L_p(\mathcal{M};\ell_2^c)}=\sum_{n\geq 0} x_n^* y_n\in L_{p/2}(\mathcal{M}).$$

Let us mention another important example of L_p-module. Let $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ be a normal conditional expectation, where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$. Then for $x,y\in L_p(\mathcal{M})$ we may consider the bracket

$$\langle x,y\rangle_{L_p(\mathcal{M};\mathcal{E})}=\mathcal{E}(x^*y)\in L_{p/2}(\mathcal{N}),$$

where $\mathcal{E}$ denotes the extension of $\mathcal{E}$ to $L_{p/2}(\mathcal{M})$ (see [24] for details on conditional expectations). It is clear that this defines an $L_{p/2}(\mathcal{N})$-valued inner product, and the associated L_p $\mathcal{N}$-module is denoted by $L_p(\mathcal{M};\mathcal{E})$. Actually, it is proved in Proposition 2.8 of [20] that this latter example is similar to the former one. More precisely, this Proposition shows that $L_p(\mathcal{M};\mathcal{E})$ is isometrically isomorphic, as a module, to a complemented subspace of $L_p(\mathcal{N};\ell_2^c)$. This result have been extended in [23] for any L_p $\mathcal{M}$-module. By Theorem 3.6 of [23], a right $\mathcal{M}$-module X is a right L_p $\mathcal{M}$-module if and only if X is a "column sum of L_p-spaces" in the following sense.

Theorem 3.1.10 ([23])**.** *Let X be a right L_p $\mathcal{M}$-module. Then X is isometrically isomorphic, as an L_p-module, to a principal L_p-module, i.e., there exists a set $(q_\alpha)_{\alpha\in I}$ of projections in $\mathcal{M}$ such that*

$$X\cong\Big\{(\xi_\alpha)_{\alpha\in I}\ :\ \xi_\alpha\in q_\alpha L_p(\mathcal{M}),\ \sum_\alpha \xi_\alpha^*\xi_\alpha\in L_{p/2}(\mathcal{M})\Big\}.$$

This latter set is denoted by $\oplus_I q_\alpha L_p(\mathcal{M})$ and endowed with the norm $\|(\xi_\alpha)_\alpha\|=\Big\|\sum_\alpha \xi_\alpha^*\xi_\alpha\Big\|_{p/2}^{1/2}$. In the finite case, if we have a projective system of L_p $\mathcal{M}$-modules in the sense of the following Corollary with some density property, then we may represent this family by using the same set of projections.

Corollary 3.1.11. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $(X_p)_{1\leq p\leq\infty}$ be a family of right $\mathcal{M}$-modules such that*

(i) X_p *is an* L_p $\mathcal{M}$*-module for all* $1\leq p\leq\infty$.

(ii) There exists a family of modular maps $I_{q,p}:X_q\to X_p$ *for* $p\leq q$ *satisfying* $I_{p,p}=id_{X_p}$ *and* $I_{q,p}\circ I_{r,q}=I_{r,p}$ *for* $p\leq q\leq r$.

(iii) The inner products are compatible with the maps $I_{q,p}$*, i.e.,*

$$\langle x,y\rangle_{X_q}=\langle I_{q,p}(x),I_{q,p}(y)\rangle_{X_p}$$

for $p\leq q$ *and* $x,y\in X_q$.

(iv) $I_{\infty,p}(X_\infty)$ *is dense in* X_p *for all* $1\leq p\leq\infty$.

Then there exists a set $(q_\alpha)_{\alpha\in I}$ of projections in $\mathcal{M}$ such that for all $1 \leq p \leq \infty$, X_p is isometrically isomorphic, as an L_p-module, to $\oplus_I q_\alpha L_p(\mathcal{M})$.

Proof. Observe that (iii) implies that the maps $I_{q,p}$ are contractive and injective. Indeed, for $p \leq q$ and $x \in X_q$, since $\mathcal{M}$ is finite we have

$$\begin{aligned}\|I_{q,p}(x)\|_{X_p} &= \|\langle I_{q,p}(x), I_{q,p}(x)\rangle_{X_p}\|_{p/2}^{1/2}\\ &= \|\langle x,x\rangle_{X_q}\|_{p/2}^{1/2} \leq \|\langle x,x\rangle_{X_q}\|_{q/2}^{1/2}\\ &= \|x\|_{X_q}.\end{aligned}$$

For the injectivity, if $I_{q,p}(x) = 0$ then $\langle I_{q,p}(x), I_{q,p}(x)\rangle_{X_p} = 0$ in $L_{p/2}(\mathcal{M})$. By (iii), this implies that $\langle x,x\rangle_{X_q} = 0$ in $L_{q/2}(\mathcal{M})$, hence $x = 0$ in X_q by (i) of Definition 3.1.9. We now turn to the proof of the Corollary. We first apply Theorem 3.1.10 to the L_∞ $\mathcal{M}$-module X_∞ and obtain a set $(q_\alpha)_{\alpha\in I}$ of projections in $\mathcal{M}$ and an isometric isomorphism of L_p-modules $\phi_\infty : X_\infty \to \oplus_I q_\alpha L_\infty(\mathcal{M})$. We may extend this isomorphism to X_p by density as follows. For $1 \leq p < \infty$ and $x = I_{\infty,p}(y) \in I_{\infty,p}(X_\infty)$ we set

$$\phi_p(x) = \phi_\infty(y) \in \oplus_I q_\alpha L_\infty(\mathcal{M}).$$

Since $\mathcal{M}$ is finite, we have a contractive inclusion $\oplus q_\alpha L_\infty(\mathcal{M}) \subset \oplus q_\alpha L_p(\mathcal{M})$ and ϕ_p preserves the $L_{p/2}(\mathcal{M})$-valued inner product. Indeed, for $x_1 = I_{\infty,p}(y_1), x_2 = I_{\infty,p}(y_2) \in I_{\infty,p}(X_\infty)$, the modularity of ϕ_∞ implies

$$\begin{aligned}\langle \phi_p(x_1), \phi_p(x_2)\rangle_{\oplus q_\alpha L_p(\mathcal{M})} &= \langle \phi_\infty(y_1), \phi_\infty(y_2)\rangle_{\oplus_I q_\alpha L_\infty(\mathcal{M})} = \langle y_1, y_2\rangle_{X_\infty}\\ &= \langle I_{\infty,p}(y_1), I_{\infty,p}(y_2)\rangle_{X_p} \quad \text{by (iii)}\\ &= \langle x_1, x_2\rangle_{X_p}.\end{aligned}$$

Hence by the density assumption (iv) we can extend ϕ_p to an isometric homomorphism of L_p-modules on X_p to $\oplus q_\alpha L_p(\mathcal{M})$. Since $\oplus q_\alpha L_\infty(\mathcal{M})$ is dense in $\oplus q_\alpha L_p(\mathcal{M})$, by the same way we can construct ϕ_p^{-1}. Thus we obtain an isometric isomorphism of L_p-modules ϕ_p which makes the following diagram commuting

$$\begin{tikzcd} X_\infty \arrow[r, leftrightarrow, "\phi_\infty"] \arrow[d, "I_{\infty,p}"'] & \oplus q_\alpha L_\infty(\mathcal{M}) \arrow[d, "id"] \\ X_p \arrow[r, leftrightarrow, "\phi_p"'] & \oplus q_\alpha L_p(\mathcal{M}) \end{tikzcd}$$

□

In this situation, we may deduce the following results from some well-known facts on the column L_p-spaces $\oplus q_\alpha L_p(\mathcal{M})$.

Corollary 3.1.12. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $(X_p)_{1\leq p\leq\infty}$ be a family of right $\mathcal{M}$-modules as in Corollary 3.1.11.*

(i) Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(X_p)^* = X_{p'}$ isometrically.

(ii) Let $1 \leq p_1 < p < p_2 \leq \infty$ and $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then

$$X_p = [X_{p_1}, X_{p_2}]_\theta.$$

Throughout all this paper, we consider a finite von Neumann algebra $\mathcal{M}$ equipped with a normal faithful normalized trace τ and we restrict ourselves to finite martingales on the interval $[0,1]$.

3.2 The $\mathcal{H}_p$-spaces

In this section we study the Hardy space $\mathcal{H}_p$ associated to the continuous filtration $(\mathcal{M}_t)_{0\leq t\leq 1}$. We start by defining the column Hardy spaces $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$. Viewing these spaces as subspaces of some ultraproduct spaces, which have an L_p-module structure, the crucial monotonicity property will imply that these two candidates for the Hardy spaces in the continuous setting are in fact equivalent. Then, introducing a third technical characterization $\widetilde{\mathcal{H}}_p^c$ of this space, we will obtain the expected duality results. We will also describe the associated $\mathcal{BMO}$ spaces and establish the analogue of the Fefferman-Stein duality in this setting. We will end this section with the study of the Hardy space $\mathcal{H}_p$ and the analogue of the Burkholder-Gundy inequalities.

3.2.1 The discrete case

Let us first recall the definitions of the Hardy spaces of noncommutative martingales in the discrete case and some well-known results. Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration of $\mathcal{M}$. Following [35], we introduce the column and row versions of square functions relative to a (finite) martingale $x=(x_n)_{n\geq 0}$:

$$S_c(x)=\Big(\sum_{n=0}^{\infty}|d_n(x)|^2\Big)^{1/2} \quad \text{and} \quad S_r(x)=\Big(\sum_{n=0}^{\infty}|d_n(x)^*|^2\Big)^{1/2},$$

where

$$d_n(x)=\begin{cases} x_n-x_{n-1} & \text{for } n\geq 1\\ x_0 & \text{for } n=0\end{cases}$$

denotes the martingale difference sequence. For $1\leq p<\infty$ we define H_p^c (resp. H_p^r) as the completion of all finite L_p-martingales under the norm $\|x\|_{H_p^c}=\|S_c(x)\|_p$ (resp. $\|x\|_{H_p^r}=\|S_r(x)\|_p$). The Hardy space of noncommutative martingales is defined by

$$H_p=\begin{cases} H_p^c+H_p^r & \text{for } \ 1\leq p<2\\ H_p^c\cap H_p^r & \text{for } \ 2\leq p<\infty\end{cases}.$$

We now recall some known facts on the column Hardy spaces. For $1\leq p<\infty$, H_p^c embeds isometrically into $L_p(\mathcal{M};\ell_2^c)$ via the map

$$i:\begin{cases} H_p^c & \longrightarrow \quad L_p(\mathcal{M};\ell_2^c)\\ x & \longmapsto \quad \sum_{n\geq 0} e_{n,0}\otimes d_n(x)\end{cases}.$$

Moreover, the Stein inequality (see [35]) implies that the map

$$\mathcal{D}:\begin{cases} L_p(\mathcal{M};\ell_2^c) & \longrightarrow \quad H_p^c\\ \sum_{n\geq 0} e_{n,0}\otimes a_n & \longmapsto \quad \sum_{n\geq 0} d_n(a_n)\end{cases}$$

is bounded for $1<p<\infty$. Here we denote $d_n(a_n)=\mathcal{E}_n(a_n)-\mathcal{E}_{n-1}(a_n)$ for $n\geq 1$, and $d_0(a_0)=\mathcal{E}_0(a_0)$.

Proposition 3.2.1. *Let $1<p<\infty$. Then H_p^c is $\sqrt{2}\gamma_p$-complemented in $L_p(\mathcal{M};\ell_2^c)$, where γ_p denotes the constant of the noncommutative Stein inequality.*

Remark 3.2.2. Recall that

$$\gamma_p \approx \max(p,p') \text{ as } p\to 1 \text{ or } p\to\infty,$$

where p' denotes the conjugate index of p.

Since $(L_p(\mathcal{M};\ell_2^c))^* = L_{p'}(\mathcal{M};\ell_2^c)$ isometrically for $\frac{1}{p}+\frac{1}{p'}=1$, we deduce the following duality result.

Corollary 3.2.3. *Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then*

$$(H_p^c)^* = H_{p'}^c \quad \textit{with equivalent norms.}$$

Moreover,

$$(\sqrt{2}\gamma_p)^{-1}\|x\|_{H_{p'}^c} \le \|x\|_{(H_p^c)^*} \le \|x\|_{H_{p'}^c}.$$

In the sequel, we will always denote the conjugate of p by p'.

Remark 3.2.4. Observe that under the above identification, for $1<p<\infty$ we have $\mathcal{D}=i^*$. Indeed, for $y\in H_p^c$ and
$a=\sum_n e_{n,0}\otimes a_n \in L_{p'}(\mathcal{M};\ell_2^c)$ we can write, by the orthogonality of martingale differences

$$\begin{aligned}(\mathcal{D}(a)|y) &= \tau\Big(\Big(\sum_n d_n(a_n)\Big)^*\Big(\sum_n d_n(y)\Big)\Big) = \sum_n \tau(d_n(a_n)^* d_n(y))\\ &= \sum_n \tau(a_n^* d_n(y)) = (a|i(y)).\end{aligned}$$

For the case $p=1$, in [35] Pisier and Xu described the dual space of H_1^c as a BMO^c-space. This analogue of the Fefferman-Stein duality has been extended by the first author and Xu in [24] to the case $1<p<2$ as follows. Recall that for $1\le p\le\infty$, we say that a sequence $(x_n)_{n\ge 0}$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if $(x_n)_{n\ge0}$ admits a factorization $x_n = a y_n b$ with $a,b\in L_{2p}(\mathcal{M})$ and $(y_n)_{n\ge0}\in\ell_\infty(L_\infty(\mathcal{M}))$. The norm of $(x_n)_{n\ge0}$ is then defined as

$$\|(x_n)_{n\ge0}\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_n = a y_n b} \|a\|_{2p}\sup_{n\ge0}\|y_n\|_\infty \|b\|_{2p}.$$

It was proved in [20, 26] that if $(x_n)_{n\ge0}$ is a positive sequence in $L_p(\mathcal{M};\ell_\infty)$, then

$$\|(x_n)_{n\ge0}\|_{L_p(\mathcal{M};\ell_\infty)} = \sup\Big\{\sum_{n\ge0}\tau(x_n y_n) : y_n\in L_{p'}^+(\mathcal{M}) \text{ and } \Big\|\sum_{n\ge0} y_n\Big\|_{p'}\le 1\Big\}.$$

The norm of $L_p(\mathcal{M};\ell_\infty)$ will be denoted by $\|\sup_n^+ x_n\|_p$. We should warn the reader that $\|\sup_n^+ x_n\|_p$ is just a notation since $\sup_n x_n$ does not take any sense in the noncommutative setting. For $2<p\le\infty$ we define

$$L_p^c MO = \{x\in L_2(\mathcal{M}) \ : \ \|x\|_{L_p^cMO}<\infty\},$$

where

$$\|x\|_{L_p^cMO} = \|\sup_{n\ge0}{}^+\mathcal{E}_n|x-x_{n-1}|^2\|_{p/2}^{1/2}.$$

Here we use the convention $x_{-1}=0$. For $p=\infty$ we denote this space by BMO^c.

Theorem 3.2.5 ([35, 24])**.** *Let $1 \le p < 2$. Then*

$$(H_p^c)^* = L_{p'}^c MO \quad \textit{with equivalent norms.}$$

Moreover,

$$\lambda_p^{-1} \|x\|_{L_{p'}^c MO} \le \|x\|_{(H_p^c)^*} \le \sqrt{2}\|x\|_{L_{p'}^c MO},$$

where λ_p remains bounded as $p \to 1$.

Combining the two previous results we obtain

Proposition 3.2.6. *Let $2 < p < \infty$. Then*

$$H_p^c = L_p^c MO \quad \textit{with equivalent norms.}$$

We end this collection of results with the analogue of the noncommutative Burkholder-Gundy inequalities proved in [35].

Theorem 3.2.7. *Let $1 < p < \infty$. Then*

$$L_p(\mathcal{M}) = H_p \quad \textit{with equivalent norms.}$$

Moreover,

$$\alpha_p^{-1}\|x\|_{H_p} \le \|x\|_p \le \beta_p \|x\|_{H_p}.$$

Remark 3.2.8. According to [25] and [40] we know that

$$\begin{aligned} \alpha_p \approx (p-1)^{-1} \text{ as } p \to 1 \quad &, \quad \alpha_p \approx p \text{ as } p \to \infty \\ \beta_p \approx 1 \text{ as } p \to 1 \quad &, \quad \beta_p \approx p \text{ as } p \to \infty. \end{aligned}$$

In particular, for $p = 1$ we have a bounded inclusion $H_1 \subset L_1(\mathcal{M})$. Throughout this paper we will always denote by $\gamma_p, \lambda_p, \alpha_p$ and β_p the constants introduced previously. We will also frequently use the noncommutative Doob inequality

$$\|{\sup_n}^+ \mathcal{E}_n(a)\|_p \le \delta_p \|a\|_p \quad \text{for } 1 < p \le \infty, a \in L_p(\mathcal{M}), a \ge 0,$$

and its dual form

$$\Big\| \sum_n \mathcal{E}_n(a_n) \Big\|_p \le \delta_p' \Big\| \sum_n a_n \Big\|_p \text{for } 1 \le p < \infty,$$

for any finite sequence $(a_n)_n$ of positive elements in $L_p(\mathcal{M})$. These inequalities were proved in [20], and we will always denote by δ_p and δ_p' respectively the constants involved there. Recall that $\delta_p' = \delta_{p'}$ for $1 \le p < \infty$. Moreover, we have

$$\delta_p \approx (p-1)^{-2} \quad \text{as } \ p \to 1 \quad , \quad \delta_p \approx 1 \quad \text{as } \ p \to \infty.$$

3.2.2 Definitions of $\widehat{\mathcal{H}}_p^c$, $\mathcal{H}_p^c$ and basic properties

We fix an ultrafilter $\mathcal{U}$ over the set of all finite partitions of the interval $[0,1]$, denoted by $\mathcal{P}_{\text{fin}}([0,1])$, such that for each finite partition σ of $[0,1]$ the set

$$U_\sigma = \{\sigma' \in \mathcal{P}_{\text{fin}}([0,1]) : \sigma \subset \sigma'\} \in \mathcal{U}.$$

Let us point out that in what follows, all considered partitions will be finite. We start by introducing a candidate for the bracket $[\cdot,\cdot]$ in the noncommutative setting. For $\sigma \in \mathcal{P}_{\text{fin}}([0,1])$ fixed and $x \in \mathcal{M}$, we define the finite bracket

$$[x,x]_\sigma = \sum_{t\in\sigma} |d_t^\sigma(x)|^2.$$

Observe that $\|[x,x]_\sigma\|_{p/2}^{1/2} = \|x\|_{H_p^c(\sigma)}$, where $H_p^c(\sigma)$ denotes the noncommutative Hardy space with respect to the discrete filtration $(\mathcal{M}_t)_{t\in\sigma}$. Hence the Burkholder-Gundy inequalities recalled in Theorem 3.2.7 and the Hölder inequality imply for each finite partition σ and $x \in \mathcal{M}$

$$\begin{array}{ccccccl} \beta_p^{-1}\|x\|_p & \leq & \|[x,x]_\sigma\|_{p/2}^{1/2} & \leq & \|x\|_2 & & \text{for } 1\leq p<2 \\ \|x\|_2 & \leq & \|[x,x]_\sigma\|_{p/2}^{1/2} & \leq & \alpha_p\|x\|_p & & \text{for } 2\leq p<\infty \end{array}. \tag{3.2.1}$$

We deduce that for $1 \leq p < \infty$, $([x,x]_\sigma)^\bullet \in L_{p/2}(\mathcal{M}_\mathcal{U})$. Indeed, we see that the family $([x,x]_\sigma)_\sigma$ is uniformly bounded in $L_{p/2}(\mathcal{M})$ and in $L_{\widetilde{p}/2}(\mathcal{M})$ for any $\widetilde{p} > \max(p,2)$ (by $\alpha_{\widetilde{p}}\|x\|_{\widetilde{p}} \leq \alpha_{\widetilde{p}}\|x\|_\infty$). Hence by Lemma 3.1.7 this means that the associated element in the ultraproduct is in the regularized part. In particular for $x \in \mathcal{M}$ and $1 \leq p < \infty$, we have $([x,x]_\sigma)^\bullet \in L_{\widetilde{p}/2}(\mathcal{M}_\mathcal{U})$ for any $\widetilde{p} > \max(p,2)$. Thus we can apply the conditional expectation $\mathcal{E}_\mathcal{U}$ to this element and set

$$[x,x]_\mathcal{U} = \mathcal{E}_\mathcal{U}(([x,x]_\sigma)^\bullet).$$

Since this bracket is in $L_{\widetilde{p}/2}(\mathcal{M})$ for any $\widetilde{p} > \max(p,2)$, it is also in $L_{p/2}(\mathcal{M})$ and we may define

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|[x,x]_\mathcal{U}\|_{p/2}^{1/2}.$$

Note that for any $\widetilde{p} \geq \max(p,2)$, this coincides with the weak-limit in $L_{\widetilde{p}/2}(\mathcal{M})$, and we can write

$$\|x\|_{\widehat{\mathcal{H}}_p^c} = \|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{1/2}.$$

This definition depends a priori on the choice of the ultrafilter $\mathcal{U}$, and we should write $\|\cdot\|_{\widehat{\mathcal{H}}_p^{c,\mathcal{U}}}$. However, we will show in the sequel that in fact this quantity does not depend on $\mathcal{U}$ up to equivalent norm. Hence for the sake of simplicity we will omit the power $\mathcal{U}$ and simply denote $\|\cdot\|_{\widehat{\mathcal{H}}_p^c}$.

We also introduce the following natural candidate for the norm of the Hardy space in the continuous setting. For $x \in \mathcal{M}$ and $1 \leq p < \infty$ we define

$$\|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \|[x,x]_\sigma\|_{p/2}^{1/2} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

The family $(\|[x,x]_\sigma\|_{p/2}^{1/2})_\sigma$ is uniformly bounded by (3.2.1), hence the limit with respect to the ultrafilter $\mathcal{U}$ exists. Taking the limit in (3.2.1) we get for $x \in \mathcal{M}$

$$\begin{array}{ccccccl} \beta_p^{-1}\|x\|_p & \leq & \|x\|_{\mathcal{H}_p^c} & \leq & \|x\|_2 & & \text{for } 1\leq p<2 \\ \|x\|_2 & \leq & \|x\|_{\mathcal{H}_p^c} & \leq & \alpha_p\|x\|_p & & \text{for } 2\leq p<\infty \end{array}. \tag{3.2.2}$$

This shows that $\|\cdot\|_{\mathcal{H}_p^c}$ defines a norm on $\mathcal{M}$. As for $\|\cdot\|_{\widehat{\mathcal{H}}_p^c}$, the norm $\|\cdot\|_{\mathcal{H}_p^c}$ depends a priori on the choice of the ultrafilter $\mathcal{U}$, but we will show that it does not (up to a constant) and hence simply denote $\|\cdot\|_{\mathcal{H}_p^c}$. Moreover, the properties of the conditional expectation $\mathcal{E}_{\mathcal{U}}$ imply the following estimates for $x \in \mathcal{M}$

$$\begin{array}{lllllllll} \beta_p^{-1}\|x\|_p & \leq & \|x\|_{\mathcal{H}_p^c} & \leq & \|x\|_{\widehat{\mathcal{H}}_p^c} & \leq & \|x\|_2 & & \text{for } 1 \leq p < 2 \\ \|x\|_2 & \leq & \|x\|_{\widehat{\mathcal{H}}_p^c} & \leq & \|x\|_{\mathcal{H}_p^c} & \leq & \alpha_p\|x\|_p & & \text{for } 2 \leq p < \infty \end{array}. \tag{3.2.3}$$

Here for $2 \leq p < \infty$ we used the contractivity of $\mathcal{E}_{\mathcal{U}}$ for the $L_{p/2}$-norm, and for $1 \leq p < 2$ we need the following well-known result due to Hansen.

Lemma 3.2.9. *Let $\mathcal{A}$ be a semifinite von Neumann algebra and $T : \mathcal{A} \to \mathcal{A}$ be a trace preserving, completely positive linear contraction. Let $0 < p \leq 1$. Then*

$$T(x^p) \leq (T(x))^p \quad \textit{and} \quad \|x\|_p \leq \|T(x)\|_p$$

for each positive element $x \in \mathcal{A}$.

Then (3.2.3) shows that $\|\cdot\|_{\widehat{\mathcal{H}}_p^c}$ defines a quasinorm on $\mathcal{M}$.

Definition 3.2.10. *Let $1 \leq p < \infty$. We define the spaces $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ as the completion of $\mathcal{M}$ with respect to the (quasi)norm $\|\cdot\|_{\widehat{\mathcal{H}}_p^c}$ and $\|\cdot\|_{\mathcal{H}_p^c}$ respectively.*

We may check that for $x \in \mathcal{M}$ and $1 \leq p < \infty$, $\langle x, x\rangle_{\widehat{\mathcal{H}}_p^c} = [x, x]_{\mathcal{U}}$ extends to an $L_{p/2}(\mathcal{M})$-valued inner product on $\widehat{\mathcal{H}}_p^c$, which endows $\widehat{\mathcal{H}}_p^c$ with an L_p $\mathcal{M}$-module structure. Hence Theorem 3.1.10 implies that $\|\cdot\|_{\widehat{\mathcal{H}}_p^c}$ is a norm for $1 \leq p < \infty$.

Remark 3.2.11. Note that thanks to (3.2.3), it suffices to consider the completion of $L_2(\mathcal{M})$ (resp. $L_p(\mathcal{M})$) for $1 \leq p < 2$ (resp. $2 \leq p < \infty$). Hence we get

$$\widehat{\mathcal{H}}_p^c = \begin{cases} \overline{L_2(\mathcal{M})}^{\|\cdot\|_{\widehat{\mathcal{H}}_p^c}} & \text{for } 1 \leq p < 2 \\ \overline{L_p(\mathcal{M})}^{\|\cdot\|_{\widehat{\mathcal{H}}_p^c}} & \text{for } 2 \leq p < \infty \end{cases} \quad \text{and} \quad \mathcal{H}_p^c = \begin{cases} \overline{L_2(\mathcal{M})}^{\|\cdot\|_{\mathcal{H}_p^c}} & \text{for } 1 \leq p < 2 \\ \overline{L_p(\mathcal{M})}^{\|\cdot\|_{\mathcal{H}_p^c}} & \text{for } 2 \leq p < \infty \end{cases}.$$

Moreover, for $1 \leq p < 2$, $L_q(\mathcal{M})$ is dense in $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ for any $q > 2$.

The crucial observation for the study of the spaces $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ is that the $H_p^c(\sigma)$-norms verify some monotonicity properties.

Lemma 3.2.12. *Let $1 \leq p < \infty$ and $\sigma \in \mathcal{P}_{\text{fin}}([0,1])$.*

(i) Let $1 \leq p \leq 2$, $x \in L_2(\mathcal{M})$ and $\sigma' \supset \sigma$. Then

$$\|x\|_{H_p^c(\sigma)} \leq \beta_p\|x\|_{H_p^c(\sigma')}.$$

Hence

$$\|x\|_{\mathcal{H}_p^c} \leq \sup_\sigma \|x\|_{H_p^c(\sigma)} \leq \beta_p\|x\|_{\mathcal{H}_p^c}.$$

(ii) Let $2 \leq p < \infty$. Let $\sigma^1, \cdots, \sigma^M$ be partitions contained in σ, let $(\alpha_m)_{1\leq m\leq M}$ be a sequence of positive numbers such that $\sum_m \alpha_m = 1$, and let $x^1, \cdots, x^M \in L_p(\mathcal{M})$. Then for $x = \sum_m \alpha_m x^m$ we have

$$\|x\|_{H_p^c(\sigma)} \leq \alpha_p\Big\|\sum_{m=1}^M \alpha_m[x^m, x^m]_{\sigma^m}\Big\|_{p/2}^{1/2}.$$

In particular for $x \in L_p(\mathcal{M})$ and $\sigma \subset \sigma'$ we have

$$\|x\|_{H_p^c(\sigma')} \leq \alpha_p \|x\|_{H_p^c(\sigma)}.$$

Hence

$$\alpha_p^{-1}\|x\|_{\mathcal{H}_p^c} \leq \inf_\sigma \|x\|_{H_p^c(\sigma)} \leq \|x\|_{\mathcal{H}_p^c}.$$

Proof. Let $1 \leq p \leq 2$, $x \in L_2(\mathcal{M})$ and $\sigma \subset \sigma'$. Applying the noncommutative Burkholder-Gundy inequalities to

$$y = \sum_{t \in \sigma} e_{t,0} \otimes d_t^\sigma(x)$$

in $L_p(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})$ for the finite partition σ', we get

$$\|y\|_{L_p(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})} \leq \beta_p \|y\|_{H_p^c(\sigma')(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})}.$$

Here we consider the discrete filtration of $B(\ell_2(\sigma))\overline{\otimes}\mathcal{M}$ given by $(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M}_t)_{t\in\sigma'}$. Note that

$$\|y\|_{H_p^c(\sigma')(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})} = \Big\| \sum_{s\in\sigma'} \sum_{t\in\sigma} e_{s,0} \otimes e_{t,0} \otimes d_s^{\sigma'}(d_t^\sigma(x)) \Big\|_{L_p(B(\ell_2(\sigma'))\overline{\otimes}B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})}.$$

An easy computation gives that for $s \in \sigma'$, $t \in \sigma$

$$d_s^{\sigma'}(d_t^\sigma(x)) = \begin{cases} d_s^{\sigma'}(x) & \text{if} \quad t^- \leq s^- < s \leq t \\ 0 & \text{otherwise} \end{cases}.$$

Hence for $s \in \sigma'$ fixed, only one term does not vanish in the sum over $t \in \sigma$ and we get

$$\|y\|_{H_p^c(\sigma')(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})} = \Big\| \sum_{s\in\sigma'} e_{s,0} \otimes d_s^{\sigma'}(x) \Big\|_{L_p(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})} = \|x\|_{H_p^c(\sigma')}.$$

The result follows from the fact that $\|y\|_{L_p(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})} = \|x\|_{H_p^c(\sigma)}$.

We now consider $2 \leq p < \infty$. Let us first assume that the partitions σ^m are disjoint. Denote σ' the union of $\sigma^1, \cdots, \sigma^M$. As above, we apply the noncommutative Burkholder-Gundy inequalities to

$$y = \sum_{m=1}^M \sum_{t\in\sigma^m} e_{t,0} \otimes \sqrt{\alpha_m}\, d_t^{\sigma^m}(x^m)$$

in $L_p(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})$ for the finite partition σ. We get

$$\|y\|_{H_p^c(\sigma)(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})} \leq \alpha_p \|y\|_{L_p(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})}.$$

On the one hand, since the partitions σ^m are disjoint we have

$$\|y\|_{L_p(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})} = \Big\| \sum_{m=1}^M \sum_{t\in\sigma^m} \alpha_m |d_t^{\sigma^m}(x^m)|^2 \Big\|_{p/2}^{1/2} = \Big\| \sum_{m=1}^M \alpha_m [x^m, x^m]_{\sigma^m} \Big\|_{p/2}^{1/2}.$$

On the other hand,

$$\|y\|_{H_p^c(\sigma)(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})} = \Big\| \sum_{s\in\sigma} \sum_{m=1}^M \sum_{t\in\sigma^m} e_{s,0} \otimes e_{t,0} \otimes \sqrt{\alpha_m}\, d_s^\sigma(d_t^{\sigma^m}(x^m)) \Big\|_{L_p(B(\ell_2(\sigma))\overline{\otimes}B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})}.$$

Again, for $s \in \sigma$ and $m \in \{1, \cdots, M\}$ fixed, since $\sigma^m \subset \sigma$, only one term does not vanish in the sum over $t \in \sigma^m$, and it is equal to $d_s^\sigma(x^m)$. Hence

$$\|y\|_{H_p^c(\sigma)(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})} = \Big\| \sum_{s\in\sigma} \sum_{m=1}^{M} \alpha_m |d_s^\sigma(x^m)|^2 \Big\|_{p/2}^{1/2}.$$

By the operator convexity of $|\cdot|^2$ we obtain

$$\|x\|_{H_p^c(\sigma)} = \Big\| \sum_{s\in\sigma} \Big| \sum_{m=1}^{M} \alpha_m d_s^\sigma(x^m) \Big|^2 \Big\|_{p/2}^{1/2} \leq \alpha_p \|y\|_{H_p^c(\sigma)(B(\ell_2(\sigma'))\overline{\otimes}\mathcal{M})},$$

which yields the required inequality. In the general case, when the partitions are not disjoint, the result still holds by approximation, thanks to the fact that the filtration is right continuous. Indeed, if there exists a common point t which is both in σ^m and σ^n (for $n \neq m$), then we can replace t by $t+\varepsilon$ in σ^m (for ε small enough), which does not change the considered norms when passing to the limit as $\varepsilon \to 0$. □

This monotonicity property immediately implies the following crucial result.

Theorem 3.2.13. *For $1 \leq p < \infty$ the space $\mathcal{H}_p^c$ is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.*

By definition, we deduce from the discrete case the following

Lemma 3.2.14. *Let $1 < p < \infty$. Then $\mathcal{H}_p^c$ is reflexive.*

Proof. It suffices to observe that the $\mathcal{H}_p^c$-norm satisfies the Clarkson inequalities. Then we will deduce that $\mathcal{H}_p^c$ is uniformly convex, so reflexive. Note that for each σ, the $H_p^c(\sigma)$-norm satisfies the Clarkson inequalities with relevant constants depending only on p. This comes from the fact that the noncommutative L_p-spaces do (see [36]), and recall that for $x \in \mathcal{M}$ we have

$$\|x\|_{H_p^c(\sigma)} = \Big\| \sum_{t\in\sigma} e_{t,0} \otimes d_t^\sigma(x) \Big\|_{L_p(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})}.$$

Taking the limit over σ yields the desired Clarkson inequalities for the $\mathcal{H}_p^c$-norm. □

3.2.3 Ultraproduct spaces and L_p-modules

In this subsection we introduce some ultraproduct spaces and their regularized versions, into which we will isometrically embed the Hardy spaces introduced in the previous subsection. We will equip these ultraproduct spaces with some L_p-module structure. One of the aim of the regularization process is to get L_p-module with respect to finite von Neumann algebras, which will allow us to deduce density results.

We first define the ultraproduct of the column L_p-spaces.

Definition 3.2.15. *Let $1 \leq p < \infty$. We define*

$$\widetilde{K}_p^c(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p(\mathcal{M}; \ell_2^c(\sigma)) \quad \textit{and} \quad K_p^c(\mathcal{U}) = \widetilde{K}_p^c(\mathcal{U}) \cdot e_{\mathcal{U}},$$

where $\cdot$ denotes the right modular action of $\widetilde{\mathcal{M}}_{\mathcal{U}}$ on $\widetilde{K}_p^c(\mathcal{U})$.
For $p = \infty$ we set

$$\widetilde{K}_\infty^c(\mathcal{U}) = \overline{\prod\nolimits_{\mathcal{U}} L_\infty(\mathcal{M}; \ell_2^c(\sigma))}^{so} \quad \textit{and} \quad K_\infty^c(\mathcal{U}) = \widetilde{K}_\infty^c(\mathcal{U}) \cdot e_{\mathcal{U}},$$

where the strong operator topology is taken in the von Neumann algebra generated by $\prod_{\mathcal{U}} B(\ell_2(\sigma))\overline{\otimes}\mathcal{M}$, *and coincides with the topology arising from the seminorms*

$$\|\xi\|_\eta = \lim_{\sigma,\mathcal{U}} \tau\Big(\eta_\sigma \sum_{t\in\sigma} |\xi_\sigma(t)|^2\Big)^{1/2}, \quad \text{for } \eta = (\eta_\sigma)^\bullet \in (\widetilde{\mathcal{M}}_{\mathcal{U}})_*^+ = \Big(\prod\nolimits_{\mathcal{U}} L_1(\mathcal{M})\Big)^+.$$

The right $\widetilde{\mathcal{M}}_{\mathcal{U}}$-module structure of $\widetilde{K}_p^c(\mathcal{U})$ is given for $x = (x_\sigma)^\bullet \in \prod_{\mathcal{U}} \mathcal{M}$ and $\xi = (\xi_\sigma)^\bullet \in \widetilde{K}_p^c(\mathcal{U})$ by

$$\xi \cdot x = (\xi_\sigma \cdot x_\sigma)^\bullet.$$

It is easy to see that this does not depend on the chosen representing families. Moreover, by Proposition 5.2 of [23], this module action extends naturally from $\prod_{\mathcal{U}} \mathcal{M}$ to $\widetilde{\mathcal{M}}_{\mathcal{U}}$. Similarly, for $\xi = (\xi_\sigma)^\bullet$, $\eta = (\eta_\sigma)^\bullet \in \widetilde{K}_p^c(\mathcal{U})$ we consider the componentwise bracket

$$\langle \xi, \eta\rangle_{\widetilde{K}_p^c(\mathcal{U})} = (\langle \xi_\sigma, \eta_\sigma\rangle_{L_p(\mathcal{M};\ell_2^c(\sigma))})^\bullet = \Big(\sum_{t\in\sigma} \xi_\sigma(t)^*\eta_\sigma(t)\Big)^\bullet \in \prod_{\mathcal{U}} L_{p/2}(\mathcal{M}) \cong L_{p/2}(\widetilde{\mathcal{M}}_{\mathcal{U}}),$$

where $\xi_\sigma = \sum_{t\in\sigma} e_{t,0}\otimes \xi_\sigma(t)$, $\eta_\sigma = \sum_{t\in\sigma} e_{t,0}\otimes\eta_\sigma(t) \in L_p(\mathcal{M};\ell_2^c(\sigma))$. This defines an $L_{p/2}(\widetilde{\mathcal{M}}_{\mathcal{U}})$-valued inner product which generates the norm of $\widetilde{K}_p^c(\mathcal{U})$ and is compatible with the module action. Hence $\widetilde{K}_p^c(\mathcal{U})$ is a right L_p $\widetilde{\mathcal{M}}_{\mathcal{U}}$-module for $1 \le p \le \infty$. The regularized spaces will play a crucial role in the sequel. We may equip $K_p^c(\mathcal{U})$ with an L_p-module structure over the finite von Neumann $\mathcal{M}_{\mathcal{U}}$ thanks to the following observation.

Lemma 3.2.16. *Let* $1 \le p \le \infty$. *Let* $\xi \in \widetilde{K}_p^c(\mathcal{U})$. *Then the following assertions are equivalent.*

(i) $\xi \in K_p^c(\mathcal{U})$;

(ii) $\langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})} \in L_{p/2}(\mathcal{M}_{\mathcal{U}})$;

Proof. By (3.1.1), it suffices to show that for $\xi \in \widetilde{K}_p^c(\mathcal{U})$ we have

$$\xi = \xi\cdot e_{\mathcal{U}} \Leftrightarrow \langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})} = \langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})} e_{\mathcal{U}}.$$

This comes from Definition 3.1.9 and the fact that $e_{\mathcal{U}}$ is a central projection. Indeed, we can write

$$\begin{aligned}
\xi = \xi\cdot e_{\mathcal{U}} &\Leftrightarrow \xi\cdot(1-e_{\mathcal{U}}) = 0\\
&\Leftrightarrow \langle \xi\cdot(1-e_{\mathcal{U}}), \xi\cdot(1-e_{\mathcal{U}})\rangle_{\widetilde{K}_p^c(\mathcal{U})} = 0\\
&\Leftrightarrow (1-e_{\mathcal{U}})^*\langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})}(1-e_{\mathcal{U}}) = 0\\
&\Leftrightarrow \langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})}(1-e_{\mathcal{U}}) = 0\\
&\Leftrightarrow \langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})} = \langle \xi,\xi\rangle_{\widetilde{K}_p^c(\mathcal{U})} e_{\mathcal{U}}.
\end{aligned}$$

□

Lemma 3.2.16 implies that $K_p^c(\mathcal{U})$ is an L_p $\mathcal{M}_{\mathcal{U}}$-module. Moreover, the family $(K_p^c(\mathcal{U}))_{1\le p\le\infty}$ forms a projective system of L_p $\mathcal{M}_{\mathcal{U}}$-modules. Indeed, for $1 \le p \le q \le \infty$ we may consider the contractive ultraproduct of the componentwise inclusion maps $I_{q,p} : \widetilde{K}_q^c(\mathcal{U}) \to \widetilde{K}_p^c(\mathcal{U})$. By modularity, this map preserves the regularized spaces, i.e., $I_{q,p} : K_q^c(\mathcal{U}) \to K_p^c(\mathcal{U})$. Then we observe that the assumptions (i)-(iii) of Corollary 3.1.11 are satisfied. In particular, we deduce that the map $I_{q,p}$ is injective on $K_q^c(\mathcal{U})$. Hence for $1 \le p \le q \le \infty$ we may identify $K_q^c(\mathcal{U})$ with a subspace of $K_p^c(\mathcal{U})$. We can prove the density assumption (iv) of Corollary 3.1.11 by using the p-equiintegrability as follows.

Lemma 3.2.17. *Let $1 \le p < \infty$. Then $K^c_\infty(\mathcal{U})$ is dense in $K^c_p(\mathcal{U})$.*

Proof. Let $\xi \in K^c_p(\mathcal{U})$, then Lemma 3.2.16 yields that $\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \in L_{p/2}(\mathcal{M}_\mathcal{U})$. Combining Theorem 3.1.6 with Lemma 3.1.5 we deduce that

$$\lim_{T\to\infty} \|\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} > T)\|_{L_{p/2}(\mathcal{M}_\mathcal{U})} = 0. \tag{3.2.4}$$

We set $\eta_T = \xi \cdot \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \le T)$. Then

$$\langle \eta_T, \eta_T \rangle_{K^c_p(\mathcal{U})} = \langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \le T) \in \mathcal{M}_\mathcal{U},$$

and $\eta_T \in K^c_\infty(\mathcal{U})$. Moreover, by (3.2.4) we have

$$\begin{aligned}
\|\xi - \eta_T\|_{K^c_p(\mathcal{U})} &= \|\xi \cdot \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} > T)\|_{K^c_p(\mathcal{U})} \\
&= \|\langle \xi \cdot \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} > T), \xi \cdot \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} > T)\rangle_{K^c_p(\mathcal{U})}\|^{1/2}_{p/2} \\
&= \|\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \mathbb{1}(\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} > T)\|^{1/2}_{p/2} \overset{T\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This ends the proof of the Lemma. □

Since $\mathcal{M}_\mathcal{U}$ is finite, we deduce duality and interpolation results from Corollary 3.1.12.

Corollary 3.2.18. *Let $1 \le p < \infty$. Then*

(i) $(K^c_p(\mathcal{U}))^* = K^c_{p'}(\mathcal{U})$ *isometrically.*

(ii) Let $1 \le p_1 < p < p_2 \le \infty$ and $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then

$$K^c_p(\mathcal{U}) = [K^c_{p_1}(\mathcal{U}), K^c_{p_2}(\mathcal{U})]_\theta \quad \textit{isometrically.}$$

(iii) $K^c_p(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{K^c_{\widetilde{p}}}(\mathcal{U}))}^{\|\cdot\|_{K^c_p(\mathcal{U})}}$.

Proof. The assertions (i) and (ii) follow directly from Corollary 3.1.12. For (iv), let $\widetilde{p} > p$ and $\xi \in I_{\widetilde{p},p}(\widetilde{K^c_{\widetilde{p}}}(\mathcal{U}))$. There exists $\eta \in \widetilde{K^c_{\widetilde{p}}}(\mathcal{U})$ such that $\xi = I_{\widetilde{p},p}(\eta)$. Then by Lemma 3.1.7 we have

$$\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} = I_{\widetilde{p},p}(\langle \eta, \eta \rangle_{\widetilde{K^c_{\widetilde{p}}}(\mathcal{U})}) \in I_{\widetilde{p},p}(L_{\widetilde{p}/2}(\widetilde{\mathcal{M}}_\mathcal{U})) \subset L_{p/2}(\mathcal{M}_\mathcal{U}),$$

and Lemma 3.2.16 yields $\xi \in K^c_p(\mathcal{U})$. Conversely, let $\xi \in K^c_p(\mathcal{U})$. Then by Lemma 3.2.17 we can approximate ξ in $K^c_p(\mathcal{U})$-norm by an element $\eta \in K^c_\infty(\mathcal{U})$, which is in $I_{\widetilde{p},p}(\widetilde{K^c_{\widetilde{p}}}(\mathcal{U}))$ for all $\widetilde{p} > p$. This concludes the proof of the Corollary. □

The characterization of the regularized space $K^c_p(\mathcal{U})$ given by the assertion (iii) in the previous Lemma will be the canonical way of defining regularized spaces throughout all this paper. Moreover, the finiteness of $\mathcal{M}_\mathcal{U}$ implies

Lemma 3.2.19. *Let $1 \le p < \infty$ and $\xi \in K^c_p(\mathcal{U})$. Then*

$$\|\xi\|_{K^c_p(\mathcal{U})} = \lim_{q\to p} \|\xi\|_{K^c_q(\mathcal{U})}.$$

Proof. For $\xi \in K^c_p(\mathcal{U})$, we have $\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})} \in L_{p/2}(\mathcal{M}_\mathcal{U})$ by Lemma 3.2.16. Since $\mathcal{M}_\mathcal{U}$ is finite we may write

$$\|\xi\|_{K^c_p(\mathcal{U})} = \|\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})}\|^{1/2}_{L_{p/2}(\mathcal{M}_\mathcal{U})} = \lim_{q\to p} \|\langle \xi, \xi \rangle_{K^c_p(\mathcal{U})}\|^{1/2}_{L_{q/2}(\mathcal{M}_\mathcal{U})} = \lim_{q\to p} \|\xi\|_{K^c_q(\mathcal{U})}.$$

□

Let us now introduce the subspaces of these ultraproduct spaces consisting of martingales.

Definition 3.2.20. *Let $1 \le p < \infty$. We define*

$$\widetilde{\mathcal{H}}_p^c(\mathcal{U}) = \prod_{\mathcal{U}} H_p^c(\sigma) \quad \text{and} \quad \mathcal{H}_p^c(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{\mathcal{H}}_{\widetilde{p}}^c(\mathcal{U}))}^{\|\cdot\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})}},$$

where $I_{\widetilde{p},p} : \widetilde{\mathcal{H}}_{\widetilde{p}}^c(\mathcal{U}) \to \widetilde{\mathcal{H}}_p^c(\mathcal{U})$ denotes the contractive ultraproduct of the componentwise inclusion maps.

Remark 3.2.21. 1. Observe that for $1 \le p < \infty$, $\mathcal{H}_p^c$ embeds isometrically into $\mathcal{H}_p^c(\mathcal{U})$. Indeed, the map $i_{\mathcal{U}} : x \in \mathcal{M} \mapsto (x)^\bullet \in \mathcal{H}_p^c(\mathcal{U})$ is isometric with respect to the norms $\|\cdot\|_{\mathcal{H}_p^c}$ and $\|\cdot\|_{\mathcal{H}_p^c(\mathcal{U})}$. By the density of $\mathcal{M}$ in $\mathcal{H}_p^c$ we can extend $i_{\mathcal{U}}$ to an isometric embedding of $\mathcal{H}_p^c$ into $\mathcal{H}_p^c(\mathcal{U})$.

2. Another crucial observation is that thanks to this regularization process, we may define $\mathcal{E}_{\mathcal{U}}$ on $\mathcal{H}_p^c(\mathcal{U})$ for $1 \le p < \infty$ as follows. Let us consider

$$J_p^c : \widetilde{\mathcal{H}}_p^c(\mathcal{U}) \to \begin{cases} \prod_{\mathcal{U}} L_p(\mathcal{M}) \cong L_p(\widetilde{\mathcal{M}}_{\mathcal{U}}) & \text{for } 1 \le p < 2 \\ \prod_{\mathcal{U}} L_2(\mathcal{M}) \cong L_2(\widetilde{\mathcal{M}}_{\mathcal{U}}) & \text{for } 2 \le p < \infty \end{cases}$$

the ultraproduct map of the componentwise inclusions. Then J_p^c is bounded of norm less than β_p for $1 \le p < 2$ by the noncommutative Burkholder-Gundy inequalities, and contractive for $2 \le p < \infty$. Then by the characterization of $L_p(\mathcal{M}_{\mathcal{U}})$ given by Lemma 3.1.7, J_p^c sends the regularized space $\mathcal{H}_p^c(\mathcal{U})$ to $L_p(\mathcal{M}_{\mathcal{U}})$ for $1 \le p < 2$, and to $L_2(\mathcal{M}_{\mathcal{U}})$ for $2 \le p < \infty$. We can now apply $\mathcal{E}_{\mathcal{U}}$ to $L_p(\mathcal{M}_{\mathcal{U}})$ for $1 \le p < 2$, and to $L_2(\mathcal{M}_{\mathcal{U}})$ for $2 \le p < \infty$. Moreover, we know that the conditional expectation $\mathcal{E}_{\mathcal{U}}$ is contractive on $L_p(\mathcal{M}_{\mathcal{U}})$ for all $1 \le p < \infty$. Hence we get a bounded map

$$\mathcal{E}_{\mathcal{U}} \circ J_p^c : \mathcal{H}_p^c(\mathcal{U}) \to \begin{cases} L_p(\mathcal{M}_{\mathcal{U}}) & \text{for } 1 \le p < 2 \\ L_2(\mathcal{M}_{\mathcal{U}}) & \text{for } 2 \le p < \infty \end{cases}.$$

For convenience, this map will be denoted again by $\mathcal{E}_{\mathcal{U}}$ in the sequel, with $\mathcal{E}_{\mathcal{U}} : \mathcal{H}_p^c(\mathcal{U}) \to L_p(\mathcal{M}_{\mathcal{U}})$ for $1 \le p < 2$ and $\mathcal{E}_{\mathcal{U}} : \mathcal{H}_p^c(\mathcal{U}) \to L_2(\mathcal{M}_{\mathcal{U}})$ for $2 \le p < \infty$. Observe that $\mathcal{E}_{\mathcal{U}}$ is faithful on $L_p(\mathcal{M}_{\mathcal{U}})$, but not necessarily on $\mathcal{H}_p^c(\mathcal{U})$.

For $1 \le p < \infty$, let us consider $i = (i_\sigma)^\bullet$, the ultraproduct map of the isometric inclusions

$$i_\sigma : \begin{cases} H_p^c(\sigma) & \longrightarrow & L_p(\mathcal{M}; \ell_2^c(\sigma)) \\ x & \longmapsto & \displaystyle\sum_{t \in \sigma} e_{t,0} \otimes d_t^\sigma(x) \end{cases}$$

and $\mathcal{D} = (\mathcal{D}_\sigma)^\bullet$, the ultraproduct map of the Stein projections

$$\mathcal{D}_\sigma : \begin{cases} L_p(\mathcal{M}; \ell_2^c(\sigma)) & \longrightarrow & H_p^c(\sigma) \\ \displaystyle\sum_{t \in \sigma} e_{t,0} \otimes a_t & \longmapsto & \sum_{t \in \sigma} d_t^\sigma(a_t) \end{cases}.$$

Then $i : \widetilde{\mathcal{H}}_p^c(\mathcal{U}) \to \widetilde{K}_p^c(\mathcal{U})$ is still isometric, and note that

$$x = \mathcal{D}(i(x)) \quad \text{for } x \in \widetilde{\mathcal{H}}_p^c(\mathcal{U}).$$

Corollary 3.2.18 (iii) yields that i and $\mathcal{D}$ preserve the regularized spaces, i.e.,

$$i : \mathcal{H}_p^c(\mathcal{U}) \to K_p^c(\mathcal{U}) \quad \text{and} \quad \mathcal{D} : K_p^c(\mathcal{U}) \to \mathcal{H}_p^c(\mathcal{U}).$$

According to the noncommutative Stein inequality, $\mathcal{D}$ is a bounded projection for $1 < p < \infty$. Hence we can state the following complementation result.

Lemma 3.2.22. *Let $1 < p < \infty$. Then $\mathcal{H}_p^c(\mathcal{U})$ is $\sqrt{2}\gamma_p$-complemented in $K_p^c(\mathcal{U})$.*

We deduce from Corollary 3.2.18 (i) the analoguous duality result for $\mathcal{H}_p^c(\mathcal{U})$.

Corollary 3.2.23. *Let $1 < p < \infty$. Then*

$$(\mathcal{H}_p^c(\mathcal{U}))^* = \mathcal{H}_{p'}^c(\mathcal{U}) \quad \textit{with equivalent norms.}$$

Moreover,

$$(\sqrt{2}\gamma_p)^{-1}\|x\|_{\mathcal{H}_{p'}^c(\mathcal{U})} \leq \|x\|_{(\mathcal{H}_p^c(\mathcal{U}))^*} \leq \|x\|_{\mathcal{H}_{p'}^c(\mathcal{U})}.$$

Recall that for $1 < p < \infty$ and each partition σ, we have $\mathcal{D}_\sigma = i_\sigma^*$. We deduce that for $1 < p < \infty$, we have $\mathcal{D} = i^*$. The following density result, based on Lemma 3.2.17, will be crucial for proving duality results in subsection 3.2.5.

Lemma 3.2.24. *Let $1 \leq p \leq 2$. Then $L_2(\mathcal{M}_\mathcal{U})$ is dense in $\mathcal{H}_p^c(\mathcal{U})$.*

Proof. Let $x \in \mathcal{H}_p^c(\mathcal{U})$. It suffices to consider $x = I_{\widetilde{p},p}(y)$ where $y \in \widetilde{\mathcal{H}}_{\widetilde{p}}^c(\mathcal{U})$ for some $\widetilde{p} > p$. Let $p < p_1 < \widetilde{p}$, then we can write $x = I_{p_1,p} \circ I_{\widetilde{p},p_1}(y) := I_{p_1,p}(z)$ where $z \in \mathcal{H}_{p_1}^c(\mathcal{U})$ with $p_1 > 1$. Hence it suffices to prove the result for $1 < p < 2$. Indeed, if for all $\varepsilon > 0$ we can find $a \in L_2(\mathcal{M}_\mathcal{U})$ such that $\|z - a\|_{\widetilde{\mathcal{H}}_{p_1}^c(\mathcal{U})} < \varepsilon$, then

$$\|x - a\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} = \|I_{p_1,p}(z - a)\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} \leq \|z - a\|_{\widetilde{\mathcal{H}}_{p_1}^c(\mathcal{U})} < \varepsilon.$$

Now let $1 < p \leq 2, x \in \mathcal{H}_p^c(\mathcal{U})$ and $\varepsilon > 0$. Then $\xi = i(x) \in K_p^c(\mathcal{U})$. By Lemma 3.2.17, for $\varepsilon > 0$ there exists $\eta \in K_\infty^c(\mathcal{U})$ such that $\|\xi - \eta\|_{\widetilde{K}_p^c(\mathcal{U})} < \varepsilon$. Hence $\eta \in K_\infty^c(\mathcal{U}) \subset K_2^c(\mathcal{U})$ and $a = \mathcal{D}(\eta) \in L_2(\mathcal{M}_\mathcal{U})$ satisfies

$$\|x - a\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} = \|\mathcal{D}(\xi) - \mathcal{D}(\eta)\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} \leq \sqrt{2}\gamma_p\|\xi - \eta\|_{\widetilde{K}_p^c(\mathcal{U})} < \sqrt{2}\gamma_p\varepsilon.$$

□

To sum up, for $1 \leq p < \infty$, $\mathcal{H}_p^c$ embeds isometrically into the L_p $\mathcal{M}_\mathcal{U}$-module $K_p^c(\mathcal{U})$ via the map

$$i \circ i_\mathcal{U} : \mathcal{H}_p^c \xrightarrow{i_\mathcal{U}} \mathcal{H}_p^c(\mathcal{U}) \xrightarrow{i} K_p^c(\mathcal{U}).$$

Hence Lemma 3.2.19 still holds true for the $\mathcal{H}_p^c$-norm.

Lemma 3.2.25. *Let $1 \leq p < \infty$ and $x \in \mathcal{M}$. Then*

$$\|x\|_{\mathcal{H}_p^c} = \lim_{q \to p} \|x\|_{\mathcal{H}_q^c}.$$

Proof. For $x \in \mathcal{M}$, $i \circ i_\mathcal{U}(x) \in K_p^c(\mathcal{U})$ and by Lemma 3.2.19 we can write

$$\|x\|_{\mathcal{H}_p^c} = \|i \circ i_\mathcal{U}(x)\|_{\widetilde{K}_p^c(\mathcal{U})} = \lim_{q \to p} \|i \circ i_\mathcal{U}(x)\|_{\widetilde{K}_q^c(\mathcal{U})} = \lim_{q \to p} \|x\|_{\mathcal{H}_q^c}.$$

□

Similarly, we can embed isometrically the space $\widehat{\mathcal{H}}_p^c$ into some L_p-module. Since we can consider $\mathcal{M}$ as a subspace of $\mathcal{M}_{\mathcal{U}}$ (via the constant families), $K_p^c(\mathcal{U})$ is in particular a right $\mathcal{M}$-module for $1 \leq p < \infty$. Let $\xi, \eta \in K_p^c(\mathcal{U})$. Lemma 3.2.16 implies that the bracket $\langle \xi, \eta \rangle_{\widetilde{K}_p^c(\mathcal{U})}$ is in $L_{p/2}(\mathcal{M}_{\mathcal{U}})$. Hence we may consider

$$\langle \xi, \eta \rangle_{\widehat{K}_p^c(\mathcal{U})} = \mathcal{E}_{\mathcal{U}}(\langle \xi, \eta \rangle_{\widetilde{K}_p^c(\mathcal{U})}) \in L_{p/2}(\mathcal{M}),$$

where $\mathcal{E}_{\mathcal{U}}$ denotes the extension of the conditional expectation $\mathcal{E}_{\mathcal{U}}$ to $L_{p/2}(\mathcal{M}_{\mathcal{U}})$. This bracket defines an $L_{p/2}(\mathcal{M})$-valued inner product on $K_p^c(\mathcal{U})$. Let us denote by

$$\|\xi\|_{\widehat{K}_p^c(\mathcal{U})} = \|\langle \xi, \xi \rangle_{\widehat{K}_p^c(\mathcal{U})}\|_{p/2}^{1/2}$$

the corresponding norm. Then we define the associated right L_p $\mathcal{M}$-module as follows.

Definition 3.2.26. *Let $1 \leq p < \infty$. We define*

$$\widehat{K}_p^c(\mathcal{U}) = \begin{cases} \overline{K_2^c(\mathcal{U})}^{\|\cdot\|_{\widehat{K}_p^c(\mathcal{U})}} & \text{for } 1 \leq p < 2 \\ \overline{K_p^c(\mathcal{U})}^{\|\cdot\|_{\widehat{K}_p^c(\mathcal{U})}} & \text{for } 2 \leq p < \infty \end{cases}.$$

The map $i \circ i_{\mathcal{U}}$ defined for $x \in \mathcal{M}$ by

$$i \circ i_{\mathcal{U}}(x) = \Big(\sum_{t \in \sigma} e_{t,0} \otimes d_t^{\sigma}(x)\Big)^{\bullet}$$

extends to an isometric map $\widehat{\mathcal{H}}_p^c \to \widehat{K}_p^c(\mathcal{U})$. Since the L_p $\mathcal{M}$-module $\widehat{K}_p^c(\mathcal{U})$ is super-reflexive, we deduce

Lemma 3.2.27. *Let $1 < p < \infty$. Then $\widehat{\mathcal{H}}_p^c$ is reflexive.*

3.2.4 $\widehat{\mathcal{H}}_p^c = \mathcal{H}_p^c$

In this subsection we show that the two candidates $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$ introduced previously for the Hardy space of noncommutative martingales with respect to the continuous filtration $(\mathcal{M}_t)_{0 \leq t \leq 1}$ actually coincide. In particular we will deduce that, up to an equivalent constant, these spaces do not depend on the choice of the ultrafilter $\mathcal{U}$.

Theorem 3.2.28. *Let $1 \leq p < \infty$. Then*

$$\mathcal{H}_p^c = \widehat{\mathcal{H}}_p^c \quad \textit{with equivalent norms.}$$

Theorem 3.2.13 yields immediately

Corollary 3.2.29. *For $1 \leq p < \infty$ the space $\widehat{\mathcal{H}}_p^c$ is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.*

For the proof of Theorem 3.2.28, we first consider the range $2 \leq p < \infty$, and we show the following complementation result.

Lemma 3.2.30. *Let $2 \leq p < \infty$. Then the map $\mathcal{E}_{\mathcal{U}} \circ \mathcal{D} : \widehat{K}_p^c(\mathcal{U}) \to \mathcal{H}_p^c$ is bounded.*

Proof. First note that since $K_p^c(\mathcal{U})$ is dense in $\widehat{K_p^c}(\mathcal{U})$, it suffices to consider

$$\xi = \Big(\sum_{t\in\sigma} e_{t,0}\otimes \xi_\sigma(t)\Big)^\bullet \in K_p^c(\mathcal{U}) \quad \text{such that} \quad \|\xi\|_{\widehat{K_p^c}(\mathcal{U})} = \|\langle \xi,\xi\rangle_{\widehat{K_p^c}(\mathcal{U})}\|_{p/2}^{1/2} \le 1.$$

Then $\mathcal{E}_\mathcal{U}\circ\mathcal{D}(\xi)$ is well-defined, and $x=\mathcal{E}_\mathcal{U}\circ\mathcal{D}(\xi)=\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$, where

$$(x_\sigma)^\bullet = \mathcal{D}(\xi) = \Big(\sum_{t\in\sigma} d_t^\sigma(\xi_\sigma(t))\Big)^\bullet \in \mathcal{H}_p^c(\mathcal{U}).$$

On the one hand, the monotonicity Lemma 3.2.12 yields for each σ

$$\|x_\sigma\|_{\mathcal{H}_p^c} \le \alpha_p \|x_\sigma\|_{H_p^c(\sigma)} \le C(p),$$

where $C(p)$ depends on $\|\xi\|_{\mathrm{K}_p^c(\mathcal{U})}$. We see that $(x_\sigma)_\sigma$ is uniformly bounded in $\mathcal{H}_p^c$, which is reflexive by Lemma 3.2.14. Thus the weak-limit in $\mathcal{H}_p^c$ exists and coincides with $\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$. Then we may approximate $\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$ by convex combinations of the x_σ's in $\mathcal{H}_p^c$-norm.
On the other hand, since $\langle\xi,\xi\rangle_{\widetilde{K_p^c}(\mathcal{U})} \in L_{p/2}(\mathcal{M}_\mathcal{U})$ for $2\le p<\infty$, we see that $\langle\xi,\xi\rangle_{\widehat{K_p^c}(\mathcal{U})} = \mathcal{E}_\mathcal{U}(\langle\xi,\xi\rangle_{\widetilde{K_p^c}(\mathcal{U})})$ coincides with the weak-limit of the elements $\langle\xi_\sigma,\xi_\sigma\rangle_{L_p(\mathcal{M};\ell_2^c(\sigma))} = \sum_{t\in\sigma}|\xi_\sigma(t)|^2$ in $L_{p/2}(\mathcal{M})$. Then, by considering the weak-limit of the elements $(x_\sigma, \sum_{t\in\sigma}|\xi_\sigma(t)|^2)$ in the space $\mathcal{H}_p^c\oplus L_{p/2}(\mathcal{M})$, for $\varepsilon>0$ we can find positive numbers $(\alpha_m)_{m=1}^M$ such that $\sum_m \alpha_m = 1$ and partitions $\sigma^1,\cdots,\sigma^M$ satisfying

$$\Big\|x - \sum_{m=1}^M \alpha_m x_{\sigma^m}\Big\|_{\mathcal{H}_p^c} < \varepsilon \quad \text{and} \quad \Big\|\langle\xi,\xi\rangle_{\widehat{K_p^c}(\mathcal{U})} - \sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m}|\xi_{\sigma^m}(t)|^2\Big\|_{p/2} < \varepsilon. \tag{3.2.5}$$

Applying Lemma 3.2.12 to x_{σ^m} and $\sigma=\bigcup_m \sigma^m$ we get

$$\Big\|\sum_{m=1}^M \alpha_m x_{\sigma^m}\Big\|_{H_p^c(\sigma)} \le \alpha_p \Big\|\sum_{m=1}^M \alpha_m [x_{\sigma^m}, x_{\sigma^m}]_{\sigma^m}\Big\|_{p/2}^{1/2}.$$

The noncommutative Stein inequality implies

$$\begin{aligned}\Big\|\sum_{m=1}^M \alpha_m [x_{\sigma^m}, x_{\sigma^m}]_{\sigma^m}\Big\|_{p/2} &= \Big\|\sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m} |d_t^{\sigma^m}(\xi_{\sigma^m}(t))|^2\Big\|_{p/2}^{1/2} \\ &\le 2\Big(\Big\|\sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m}|\mathcal{E}_t(\xi_{\sigma^m}(t))|^2\Big\|_{p/2} + \Big\|\sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m}|\mathcal{E}_{t^-}(\xi_{\sigma^m}(t))|^2\Big\|_{p/2}\Big) \\ &\le 4\gamma_p^2 \Big\|\sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m}|\xi_{\sigma^m}(t)|^2\Big\|_{p/2}.\end{aligned}$$

Hence by (3.2.5) we obtain

$$\begin{aligned}\|x\|_{\mathcal{H}_p^c} &\le \varepsilon + \Big\|\sum_{m=1}^M \alpha_m x_{\sigma^m}\Big\|_{\mathcal{H}_p^c} \le \varepsilon + \alpha_p \Big\|\sum_{m=1}^M \alpha_m x_{\sigma^m}\Big\|_{H_p^c(\sigma)} \\ &\le \varepsilon + 2\gamma_p\alpha_p^2 \Big\|\sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m}|\xi_{\sigma^m}(t)|^2\Big\|_{p/2}^{1/2} \le \varepsilon + 2\gamma_p\alpha_p^2(\varepsilon + \|\xi\|^2_{\widehat{K_p^c}(\mathcal{U})})^{1/2}.\end{aligned}$$

Sending ε to 0 ends the proof. $\square$

Proof of Theorem 3.2.28 for $2 \leq p < \infty$. This is a direct consequence of Lemma 3.2.30. Indeed, it suffices to show that the $\mathcal{H}_p^c$-norm and the $\widehat{\mathcal{H}}_p^c$-norm are equivalent on $L_p(\mathcal{M})$. Let $x \in L_p(\mathcal{M})$, by (3.2.3) we have $\|x\|_{\widehat{\mathcal{H}}_p^c} \leq \|x\|_{\mathcal{H}_p^c}$. To prove the reverse inequality, we write $x = \mathcal{E}_{\mathcal{U}} \circ \mathcal{D} \circ i \circ i_{\mathcal{U}}(x)$ and Lemma 3.2.30 yields

$$\|x\|_{\mathcal{H}_p^c} = \|\mathcal{E}_{\mathcal{U}} \circ \mathcal{D}(i \circ i_{\mathcal{U}}(x))\|_{\mathcal{H}_p^c} \leq 2\gamma_p \alpha_p^2 \|i \circ i_{\mathcal{U}}(x)\|_{\widehat{K}_p^c(\mathcal{U})} = 2\gamma_p \alpha_p^2 \|x\|_{\widehat{\mathcal{H}}_p^c}.$$

□

For $1 \leq p < 2$, we will use a dual approach. The trick is to embed $\widehat{\mathcal{H}}_p^c$ into a larger ultraproduct space defined as follows. Let us fix $q > 2$ and in the sequel we will consider $\widehat{\mathcal{H}}_p^c$ as the completion of $L_q(\mathcal{M})$. We define the set

$$\mathcal{I} = \mathcal{P}_{\text{fin}}(L_q(\mathcal{M})) \times \mathcal{P}_{\text{fin}}([0,1]) \times \mathbb{R}_+^*,$$

where $\mathcal{P}_{\text{fin}}(L_q(\mathcal{M}))$ denotes the set of all finite families in $L_q(\mathcal{M})$. Then $\mathcal{I}$ is a partially ordered set by the natural order. We define an ultrafilter $\mathcal{V}$ on $\mathcal{I}$ as follows. For $G \in \mathcal{P}_{\text{fin}}(L_q(\mathcal{M}))$ we define

$$S_G = \{F \in \mathcal{P}_{\text{fin}}(L_q(\mathcal{M})) \ : \ G \subseteq F\}$$

and consider the filter base on $\mathcal{P}_{\text{fin}}(L_q(\mathcal{M}))$

$$\mathcal{T} = \{S_G \ : \ G \in \mathcal{P}_{\text{fin}}(L_q(\mathcal{M}))\}.$$

On $\mathbb{R}_+^*$ we consider the filter base given by

$$\mathcal{W} = \{\ ]0,\delta] \ : \ \delta > 0\}.$$

Then the product $\mathcal{V}' = \mathcal{T} \times \mathcal{U} \times \mathcal{W}$ is a filter base on $\mathcal{I}$, and we consider $\mathcal{V}$ an ultrafilter on $\mathcal{I}$ refining $\mathcal{V}'$. Let us now fix an element $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$. For each $x \in F$, the Burkholder-Gundy inequalities applied to each σ for $q > 2$ yields that the family $([x,x]_\sigma)_\sigma$ is uniformly bounded in $L_{q/2}(\mathcal{M})$. Since $L_{q/2}(\mathcal{M})$ is reflexive, the weak-limit exists and

$$\mathcal{E}_{\mathcal{U}}(([x,x]_\sigma)^\bullet) = w\text{-}\lim_{\sigma,\mathcal{U}} [x,x]_\sigma \quad \text{in } L_{q/2}(\mathcal{M}).$$

The same holds for the finite family F, i.e., the family $([x,x]_\sigma)_{x\in F}$ is uniformly bounded in $L_{q/2}(\mathcal{M}) \oplus \cdots \oplus L_{q/2}(\mathcal{M})$. By reflexivity, the weak-limit exists and can be approximated by convex combinations in $L_{q/2}$-norm. Hence we can find a sequence of positive numbers $(\alpha_m(i))_{m=1}^{M(i)}$ such that $\sum_m \alpha_m = 1$ and finite partitions $\sigma_i^1, \cdots, \sigma_i^{M(i)}$ satisfying for all $x \in F$

$$\Big\| w\text{-}\lim_{\sigma,\mathcal{U}} [x,x]_\sigma - \sum_{m=1}^{M(i)} \alpha_m(i) [x,x]_{\sigma_i^m} \Big\|_{q/2} < \varepsilon. \tag{3.2.6}$$

We may assume in addition that σ_i is contained in σ_i^m for all m. We consider the Hilbert space $\mathcal{H}_i = \ell_2\Big(\bigcup_{m,t\in\sigma_i^m} \{t\}\Big)$ equipped with the norm

$$\|(\xi_{m,t})_{1\leq m\leq M(i), t\in\sigma_i^m}\|_{\mathcal{H}_i} = \Big(\sum_{m=1}^{M(i)} \alpha_m(i) \sum_{t\in\sigma_i^m} |\xi_{m,t}|^2\Big)^{1/2}.$$

For $1 \le p \le \infty$ and $i \in \mathcal{I}$ we consider the column space $L_p(\mathcal{M};\mathcal{H}_i^c)$. Recall that for any sequence $(\xi_{m,t})_{1\le m\le M(i), t\in\sigma_i^m}$ in $L_p(\mathcal{M})$ we have

$$\Big\|\sum_{m=1}^{M(i)}\sum_{t\in\sigma_i^m} e_{m,0}\otimes e_{t,0}\otimes \xi_{m,t}\Big\|_{L_p(\mathcal{M};\mathcal{H}_i^c)} = \Big\|\Big(\sum_{m=1}^{M(i)}\alpha_m(i)\sum_{t\in\sigma_i^m}|\xi_{m,t}|^2\Big)^{1/2}\Big\|_p.$$

Then for $1 \le p < \infty$ we have

$$(L_p(\mathcal{M};\mathcal{H}_i^c))^* = L_{p'}(\mathcal{M};\mathcal{H}_i^c) \quad \text{isometrically,}$$

via the duality bracket

$$(\xi|\eta)_{L_p(\mathcal{M};\mathcal{H}_i^c),L_{p'}(\mathcal{M};\mathcal{H}_i^c)} = \sum_{m=1}^{M(i)}\sum_{t\in\sigma_i^m}\alpha_m(i)\tau(\xi_{m,t}^*\eta_{m,t}).$$

Lemma 3.2.31. *Let $1 \le p < 2$. Then $\widehat{\mathcal{H}}_p^c$ embeds isometrically into $\prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)$.*

Proof. By the density of $L_q(\mathcal{M})$ in $\widehat{\mathcal{H}}_p^c$, it suffices to consider an element $x \in L_q(\mathcal{M})$. We associate x with $\widetilde{x} = (\widetilde{x}(i))^\bullet \in \prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)$ defined as follows. For each index $i = (F,\sigma_i,\varepsilon) \in \mathcal{I}$ such that $x \in F$ we set

$$\widetilde{x}(i) = \sum_{m=1}^{M(i)}\sum_{t\in\sigma_i^m} e_{m,0}\otimes e_{t,0}\otimes d_t^{\sigma_i^m}(x),$$

and $\widetilde{x}(i) = 0$ otherwise. Then we claim that

$$\|\widetilde{x}\|_{\prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)} = \lim_{i,\mathcal{V}}\|\widetilde{x}(i)\|_{L_p(\mathcal{M};\mathcal{H}_i^c)} = \|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{1/2} = \|x\|_{\widehat{\mathcal{H}}_p^c}. \tag{3.2.7}$$

Indeed, for $\delta > 0$, we observe that for $i = (F,\sigma_i,\varepsilon)$ such that $x \in F$ and $\varepsilon^{p/2} \le \delta$ we have by the triangle inequality applied to the norm $\|\cdot\|_{p/2}^{p/2}$ and (3.2.6)

$$\begin{aligned}
\Big|\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{p/2} - \|\widetilde{x}(i)\|_{L_p(\mathcal{M};\mathcal{H}_i^c)}^p\Big| &= \Big|\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{p/2} - \Big\|\sum_{m=1}^{M(i)}\alpha_m(i)\sum_{t\in\sigma_i^m}|d_t^{\sigma_i^m}(x)|^2\Big\|_{p/2}^{p/2}\Big| \\
&= \Big|\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{p/2} - \Big\|\sum_{m=1}^{M(i)}\alpha_m(i)[x,x]_{\sigma_i^m}\Big\|_{p/2}^{p/2}\Big| \\
&\le \Big\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma - \sum_{m=1}^{M(i)}\alpha_m(i)[x,x]_{\sigma_i^m}\Big\|_{p/2}^{p/2} \\
&\le \Big\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma - \sum_{m=1}^{M(i)}\alpha_m(i)[x,x]_{\sigma_i^m}\Big\|_{q/2}^{p/2} \\
&< \varepsilon^{p/2} \le \delta.
\end{aligned}$$

This means that

$$S_{\{x\}}\times\mathcal{P}_{\text{fin}}([0,1])\times]0,\delta^{2/p}] \subset \{i\in\mathcal{I} \;:\; \Big|\|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|_{p/2}^{p/2} - \|\widetilde{x}(i)\|_{L_p(\mathcal{M};\mathcal{H}_i^c)}^p\Big| < \delta\}.$$

Since by construction, the set $S_{\{x\}} \times \mathcal{P}_{\text{fin}}([0,1]) \times]0, \delta^{2/p}] \in \mathcal{T} \times \mathcal{U} \times \mathcal{W}$ is in the ultrafilter $\mathcal{V}$, we deduce that the set in the right hand side is also in $\mathcal{V}$ for all $\delta > 0$. Thus by the definition of the limit with respect to an ultrafilter we get

$$\lim_{i,\mathcal{V}} \|\widetilde{x}(i)\|^p_{L_p(\mathcal{M};\mathcal{H}_i^c)} = \|w\text{-}\lim_{\sigma,\mathcal{U}}[x,x]_\sigma\|^{p/2}_{p/2}.$$

This concludes the proof of (3.2.7) and shows that the map $x \in L_q(\mathcal{M}) \mapsto \widetilde{x}$ extends to an isometric embedding of $\widehat{\mathcal{H}}_p^c$ into $\prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)$. ☐

This embedding will be useful to describe the dual space of $\widehat{\mathcal{H}}_p^c$.

Lemma 3.2.32. *Let $1 \leq p < 2$. Then*

$$(\widehat{\mathcal{H}}_p^c)^* \subset (\mathcal{H}_p^c)^*.$$

Proof. Let $\varphi \in (\widehat{\mathcal{H}}_p^c)^*$ be a functional of norm less than one. By Lemma 3.2.31 and the Hahn-Banach Theorem we can extend φ to a linear functional on $\prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)$ of norm less than one, also denoted by φ. Lemma 3.1.3 implies that φ is the weak*-limit of elements ξ_λ in the unit ball of $\prod_{\mathcal{V}}(L_p(\mathcal{M};\mathcal{H}_i^c))^* = \prod_{\mathcal{V}} L_{p'}(\mathcal{M};\mathcal{H}_i^c)$. For each λ, we will prove that there exists $z_\lambda \in L_2(\mathcal{M})$ such that

$$(\xi_\lambda|\widetilde{x}) = \tau(z_\lambda^* x),\ \forall\, x \in L_q(\mathcal{M}) \quad \text{and} \quad \|z_\lambda\|_{(\mathcal{H}_p^c)^*} \leq k_p,$$

where $\widetilde{x}$ denotes the element in $\prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)$ corresponding to x via the embedding given by Lemma 3.2.31. Then we will set $z = w\text{-}\lim_\lambda z_\lambda$, where the weak-limit is taken in $L_2(\mathcal{M})$ and we will get an element $z \in L_2(\mathcal{M})$ such that

$$\varphi(x) = \lim_\lambda (\xi_\lambda|\widetilde{x}) = \lim_\lambda \tau(z_\lambda^* x) = \tau(z^*x),\ \forall\, x \in L_q(\mathcal{M}) \quad \text{and} \quad \|z\|_{(\mathcal{H}_p^c)^*} \leq k_p.$$

Finally we will conclude the proof using the density of $L_q(\mathcal{M})$ in $\widehat{\mathcal{H}}_p^c$.

We now consider an element $\xi = (\xi(i))^\bullet \in \prod_{\mathcal{V}} L_{p'}(\mathcal{M};\mathcal{H}_i^c)$ of norm less than one, with

$$\xi(i) = \sum_{m=1}^{M(i)} \sum_{t \in \sigma_i^m} e_{m,0} \otimes e_{t,0} \otimes \xi_{m,t}(i).$$

Fix $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$ and $1 \leq m \leq M(i)$. Then $\xi_m(i) := \sum_{t \in \sigma_i^m} e_{m,0} \otimes e_{t,0} \otimes \xi_{m,t}(i) \in L_{p'}(\mathcal{M};\ell_2^c(\sigma_i^m))$. We set

$$z_m(i) = \mathcal{D}_{\sigma_i^m}(\xi_m(i)).$$

Note that since the partition σ_i^m is finite, we have $z_m(i) \in L_{p'}(\mathcal{M})$. Then we consider

$$z(i) = \sum_m \alpha_m(i) z_m(i) \in L_{p'}(\mathcal{M}).$$

We first show that $\|z(i)\|_{L^c_{p'}MO(\sigma_i')} \leq k_p$ for $\sigma_i' = \sigma_i^1 \cup \cdots \cup \sigma_i^{M(i)}$. Let $s \in \sigma_i'$. Then for m fixed, we denote by $t_m(s)$ the unique element in σ_i^m satisfying $t_m(s)^- \leq s^- < s \leq t_m(s)$. The operator convexity of the square function $|\cdot|^2$ yields

$$\begin{aligned}\mathcal{E}_s|z(i) - \mathcal{E}_{s^-}(z(i))|^2 &= \mathcal{E}_s\Big|\sum_m \alpha_m(i)(z_m(i) - \mathcal{E}_{s^-}(z_m(i)))\Big|^2 \\ &\leq \sum_m \alpha_m(i)\mathcal{E}_s|z_m(i) - \mathcal{E}_{s^-}(z_m(i))|^2.\end{aligned} \tag{3.2.8}$$

On the other hand we can write

$$\begin{aligned}
&\mathcal{E}_s|z_m(i) - \mathcal{E}_{s^-}(z_m(i))|^2 \\
&= \mathcal{E}_s\Big(\sum_{t>t_m(s), t\in\sigma_i^m} |d_t^{\sigma_i^m}(z_m(i))|^2 + |\mathcal{E}_{t_m(s)}(z_m(i)) - \mathcal{E}_{s^-}(z_m(i))|^2\Big) \\
&= \mathcal{E}_s\Big(\sum_{t>t_m(s), t\in\sigma_i^m} |d_t^{\sigma_i^m}(\xi_{m,t}(i))|^2 + |\mathcal{E}_{t_m(s)}(\xi_{m,t_m(s)}(i)) - \mathcal{E}_{s^-}(\xi_{m,t_m(s)}(i))|^2\Big) \\
&\leq 4\mathcal{E}_s\Big(\sum_{t>t_m(s), t\in\sigma_i^m} |\xi_{m,t}(i)|^2\Big) + 2\mathcal{E}_s|\xi_{m,t_m(s)}(i)|^2 + 2\mathcal{E}_{s^-}|\xi_{m,t_m(s)}(i)|^2 \\
&\leq 4\mathcal{E}_s\Big(\sum_{t\in\sigma_i^m} |\xi_{m,t}(i)|^2\Big) + 2\mathcal{E}_{s^-}\Big(\sum_{t\in\sigma_i^m} |\xi_{m,t}(i)|^2\Big).
\end{aligned}$$

Then (3.2.8) gives

$$\mathcal{E}_s|z(i) - \mathcal{E}_{s^-}(z(i))|^2 \leq 4\mathcal{E}_s\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big) + 2\mathcal{E}_{s^-}\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big).$$

By the noncommutative Doob inequality we obtain

$$\begin{aligned}
&\|z(i)\|^2_{L^c_{p'}MO(\sigma_i')} = \|{\sup_{s\in\sigma_i'}}^+ \mathcal{E}_s|z(i) - \mathcal{E}_{s^-}(z(i))|^2\|_{p'/2} \\
&\leq 4\Big\|{\sup_{s\in\sigma_i'}}^+ \mathcal{E}_s\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big)\Big\|_{p'/2} + 2\Big\|{\sup_{s\in\sigma_i'}}^+ \mathcal{E}_{s^-}\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big)\Big\|_{p'/2} \\
&\leq 6\delta_{p'/2}\Big\|\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big\|_{p'/2} = 6\delta_{p'/2}\|\xi(i)\|^2_{L_{p'}(\mathcal{M};\mathcal{H}_i^c)}.
\end{aligned}$$

Hence

$$\|z(i)\|_{L^c_{p'}MO(\sigma_i')} \leq 3\delta^{1/2}_{p'/2}\|\xi(i)\|_{L_{p'}(\mathcal{M};\mathcal{H}_i^c)} \leq 3\delta^{1/2}_{p'/2}. \tag{3.2.9}$$

In particular, we see that the family $(z(i))_i$ is uniformly bounded in $L_2(\mathcal{M})$. We set $z = w\text{-}\lim_{i,\mathcal{V}} z(i)$ in $L_2(\mathcal{M})$. By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_p^c$ we have

$$\|z\|_{(\mathcal{H}_p^c)^*} = \sup_{x\in L_2(\mathcal{M}), \|x\|_{\mathcal{H}_p^c}\leq 1} |\tau(z^*y)|.$$

Then for $x \in L_2(\mathcal{M})$, $\|x\|_{\mathcal{H}_p^c} \leq 1$, Lemma 3.2.12 and (3.2.9) imply

$$\begin{aligned}
|\tau(z^*x)| &\leq \lim_{i,\mathcal{V}} |\tau(z(i)^*x)| \leq \sqrt{2}\lim_{i,\mathcal{V}} \|z(i)\|_{L^c_{p'}MO(\sigma_i')}\|x\|_{H_p^c(\sigma_i')} \\
&\leq 3\sqrt{2}\delta^{1/2}_{p'/2}\beta_p\|x\|_{\mathcal{H}_p^c} \leq 3\sqrt{2}\delta^{1/2}_{p'/2}\beta_p.
\end{aligned}$$

Hence we get $\|z\|_{(\mathcal{H}_p^c)^*} \leq k_p$ with $k_p = 3\sqrt{2}\delta^{1/2}_{p'/2}\beta_p$. Finally, it remains to check that for all $x \in L_q(\mathcal{M})$, z satisfies

$$(\xi|\widetilde{x})_{\prod_\mathcal{V} L_{p'}(\mathcal{M};\mathcal{H}_i^c), \prod_\mathcal{V} L_p(\mathcal{M};\mathcal{H}_i^c)} = \tau(z^*x). \tag{3.2.10}$$

We first verify that for each $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$ such that $x \in F$ we have

$$(\xi(i)|\widetilde{x}(i))_{L_{p'}(\mathcal{M};\mathcal{H}_i^c), L_p(\mathcal{M};\mathcal{H}_i^c)} = \tau(z(i)^*x).$$

For each m, Remark 3.2.4 implies

$$\tau(z_m(i)^*x) = \tau(\mathcal{D}_{\sigma_i^m}(\xi_m(i))^*x) = \tau(\xi_m(i)^* i_{\sigma_i^m}(x)) = \sum_{t\in\sigma_i^m} \tau\big(\xi_{m,t}(i)^* d_t^{\sigma_i^m}(x)\big).$$

Then

$$\tau(z(i)^*x) = \sum_{m=1}^{M(i)} \alpha_m(i)\tau(z_m(i)^*x) = \sum_{m=1}^{M(i)}\sum_{t\in\sigma_i^m} \alpha_m(i)\tau\big(\xi_{m,t}(i)^* d_t^{\sigma_i^m}(x)\big) = (\xi(i)|\widetilde{x}(i)).$$

By the construction of the ultrafilter $\mathcal{V}$ this is sufficient to show that the limits along $\mathcal{V}$ coincide, and (3.2.10) follows. This concludes the proof of the Lemma. $\square$

Proof of Theorem 3.2.28 for $1 \leq p < 2$. By density, it suffices to prove the equivalence of the norms on $L_2(\mathcal{M})$. This follows from (3.2.3), and we prove the reverse inequality by duality by using Lemma 3.2.32. $\square$

In the sequel, we will use the definition of $\mathcal{H}_p^c$ to transfer the results from the discrete case to the continuous setting. Indeed, this construction seems more natural for taking the limit in the classical results. However, the duality results and the noncommutative Burkholder-Gundy inequalities do not follow immediately from the definition. In particular, it is not clear a priori that the Hardy spaces introduced previously embed into some L_p-space.

3.2.5 Duality results

The aim of this subsection is to obtain the analoguous result of Theorem 3.2.3 in the continuous setting. In particular, thanks to the definition of $\mathcal{H}_p^c$, this will imply that $\mathcal{H}_p^c$ embeds into $L_p(\mathcal{M})$ for $1 < p < 2$ and into $L_2(\mathcal{M})$ for $2 \leq p < \infty$. In fact, this also holds true for $p = 1$. In fact, for $1 \leq p < 2$ this is a direct consequence of the Lemma 3.2.12.

Proposition 3.2.33. *Let* $1 \leq p < 2$ *and* $X_p = \{x \in L_p(\mathcal{M}) \ : \ \|x\|_{\mathcal{H}_p^c} < \infty\}$. *We equip* X_p *with the norm* $\|\cdot\|_{\mathcal{H}_p^c}$. *Then*

(i) X_p *is complete.*

(ii) $\mathcal{H}_p^c$ *embeds into* $L_p(\mathcal{M})$.

(iii) For $1 < p < 2$, $X_p = (\mathcal{H}_{p'}^c)^*$ *with equivalent norms.*

Proof. The argument we will use to prove the completeness of the space X_p relies on the fact that the discrete $H_p^c(\sigma)$-norms are increasing in σ (up to a constant) for $1 \leq p < 2$, and on the completeness of the discrete spaces $H_p^c(\sigma)$. Let $(x_n)_{n\geq 1} \subset X_p$ be a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{H}_p^c}$. Recall that for $x \in X_p$ we have $\|x\|_p \leq \beta_p \|x\|_{\mathcal{H}_p^c}$. Then we deduce that $(x_n)_{n\geq 1}$ is also a Cauchy sequence in $L_p(\mathcal{M})$. Hence $(x_n)_{n\geq 1}$ converges in $L_p(\mathcal{M})$ to an element $x \in L_p(\mathcal{M})$. Since for a finite partition σ, the norms $\|\cdot\|_p$ and $\|\cdot\|_{H_p^c(\sigma)}$ are equivalent, the convergence is in $H_p^c(\sigma)$ for each σ. It remains to prove that the convergence is also with respect to the $\mathcal{H}_p^c$-norm, and then we will conclude that $x \in X_p$. Fix $\varepsilon > 0$. By the Cauchy property with respect to the $\mathcal{H}_p^c$-norm, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\lim_{m\to\infty} \|x_m - x_n\|_{\mathcal{H}_p^c} < \varepsilon.$$

For a fixed partition σ, since $x_n \to x$ in $H_p^c(\sigma)$ we have

$$\|x - x_n\|_{H_p^c(\sigma)} = \lim_{m\to\infty} \|x_m - x_n\|_{H_p^c(\sigma)} \leq \beta_p \lim_{m\to\infty} \|x_m - x_n\|_{\mathcal{H}_p^c} < \varepsilon.$$

Note that here n_0 does not depend on the partition σ, hence taking the limit in σ we obtain the required convergence in $\mathcal{H}_p^c$-norm.

The embedding of $\mathcal{H}_p^c$ into $L_p(\mathcal{M})$ follows directly from the previous point. Indeed, we can now isometrically embed $\mathcal{H}_p^c$ into X_p, which is a subspace of $L_p(\mathcal{M})$.

We turn to assertion (iii). Let $1 < p < 2$ and $x \in X_p$. For $y \in L_{p'}(\mathcal{M})$ and a fixed partition σ, the Hölder inequality implies

$$|\tau(x^*y)| = \Big|\sum_{t\in\sigma} \tau(d_t^\sigma(x)^* d_t^\sigma(y))\Big| \leq \Big\|\Big(\sum_{t\in\sigma} |d_t^\sigma(x)|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{t\in\sigma} |d_t^\sigma(y)|^2\Big)^{1/2}\Big\|_{p'} = \|x\|_{H_p^c(\sigma)}\|y\|_{H_{p'}^c(\sigma)}.$$

Passing to the limit yields

$$|\tau(x^*y)| \leq \|x\|_{\mathcal{H}_p^c}\|y\|_{\mathcal{H}_{p'}^c}.$$

Since $L_{p'}(\mathcal{M})$ is dense in $\mathcal{H}_{p'}^c$, this shows that $x \in (\mathcal{H}_{p'}^c)^*$ and

$$\|x\|_{(\mathcal{H}_{p'}^c)^*} \leq \|x\|_{\mathcal{H}_p^c}.$$

Conversely, let $\varphi \in (\mathcal{H}_{p'}^c)^*$ be of norm less than one. Since $L_{p'}(\mathcal{M})$ is dense in $\mathcal{H}_{p'}^c$, φ is represented by an element $x \in L_p(\mathcal{M})$ such that $\varphi(y) = \tau(x^*y)$ for all $y \in L_{p'}(\mathcal{M})$. It remains to show that $\|x\|_{\mathcal{H}_p^c} < \infty$. For a fixed partition σ, by Corollary 3.2.3 and the density of $L_{p'}(\mathcal{M})$ in $H_{p'}^c(\sigma)$, we get

$$\|x\|_{H_p^c(\sigma)} \leq \sqrt{2}\gamma_p \|x\|_{(H_{p'}^c(\sigma))^*} = \sqrt{2}\gamma_p \sup_{y\in L_{p'}(\mathcal{M}), \|y\|_{H_{p'}^c(\sigma)}\leq 1} |\tau(x^*y)|.$$

For $y \in L_{p'}(\mathcal{M})$ with $\|y\|_{H_{p'}^c(\sigma)} \leq 1$ we have

$$|\tau(x^*y)| = |\varphi(y)| \leq \|y\|_{\mathcal{H}_{p'}^c} \leq \alpha_{p'}\|y\|_{H_{p'}^c(\sigma)} \leq \alpha_{p'}.$$

Hence we get

$$\|x\|_{\mathcal{H}_p^c} \leq \sqrt{2}\gamma_p\alpha_{p'}\|x\|_{(\mathcal{H}_{p'}^c)^*},$$

and deduce that $x \in X_p$. $\square$

Since we may consider $\mathcal{H}_p^c$ as a subspace of $\mathcal{H}_p^c(\mathcal{U})$, another natural way of describing the dual space of $\mathcal{H}_p^c$ is to introduce the following quotient space.

Definition 3.2.34. *Let $1 \leq p < \infty$. We define the space $\widetilde{\mathcal{H}}_p^c$ as the quotient space of $\mathcal{H}_p^c(\mathcal{U})$ by the kernel of the map $\mathcal{E}_\mathcal{U}$. The norm in $\widetilde{\mathcal{H}}_p^c$ is given by the usual quotient norm*

$$\|x\|_{\widetilde{\mathcal{H}}_p^c} = \inf_{x=\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)} \|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} = \inf_{x=\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)} \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{H_p^c(\sigma)}.$$

Recall that by the discussion following the definition of the spaces $\widetilde{\mathcal{H}}_p^c(\mathcal{U})$ (see Remark 3.2.21), for $1 \leq p < \infty$ we may define the map $\mathcal{E}_\mathcal{U}$ on the Banach space $\mathcal{H}_p^c(\mathcal{U})$. Since this map is bounded from $\mathcal{H}_p^c(\mathcal{U})$ to $L_p(\mathcal{M})$ for $1 \leq p < 2$, and to $L_2(\mathcal{M})$ for $2 \leq p < \infty$, it is clear that $(\widetilde{\mathcal{H}}_p^c, \|\cdot\|_{\widetilde{\mathcal{H}}_p^c})$ is a Banach space.

Remark 3.2.35. 1. Recall that for $1 \leq p < \infty$, the conditional expectation $\mathcal{E}_\mathcal{U}$ on $L_p(\mathcal{M}_\mathcal{U})$ coincides with the weak-limit in $L_p(\mathcal{M})$. Hence we have

$$\widetilde{\mathcal{H}}_p^c = \{w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma \ : \ (x_\sigma)^\bullet \in \mathcal{H}_p^c(\mathcal{U})\},$$

where the weak-limit is taken in $L_p(\mathcal{M})$ if $1 \leq p < 2$ and in $L_2(\mathcal{M})$ if $2 \leq p < \infty$.

2. By definition, $\widetilde{\mathcal{H}}_p^c$ embeds into $L_p(\mathcal{M})$ for $1 \leq p < 2$ and into $L_2(\mathcal{M})$ for $2 \leq p < \infty$. Hence $\widetilde{\mathcal{H}}_p^c$ is a subspace of $L_p(\mathcal{M})$ for $1 \leq p < 2$ and of $L_2(\mathcal{M})$ for $2 \leq p < \infty$ via the identification

$$\widetilde{\mathcal{H}}_p^c = \mathcal{H}_p^c(\mathcal{U})/\ker \mathcal{E}_\mathcal{U} \cong \mathcal{E}_\mathcal{U}(\mathcal{H}_p^c(\mathcal{U})).$$

Since for $1 \leq p < 2$ we may consider $L_p(\mathcal{M})$ as a subspace of $\mathcal{H}_p^c(\mathcal{U})$ via the map $i_\mathcal{U}$, and similarly for $2 \leq p < \infty$ we can see $L_2(\mathcal{M})$ as a subspace of $\mathcal{H}_p^c(\mathcal{U})$, the previous identification allows us to consider $\widetilde{\mathcal{H}}_p^c$ as a complemented subspace of $\mathcal{H}_p^c(\mathcal{U})$ for $1 \leq p < \infty$.

This space will be a crucial tool for proving our main result. Indeed, it describes naturally the dual space of $\mathcal{H}_p^c$.

Proposition 3.2.36. *Let $1 < p < \infty$. Then*

$$(\mathcal{H}_p^c)^* = \widetilde{\mathcal{H}}_{p'}^c \quad \textit{with equivalent norms.}$$

Proof. Let $x \in \widetilde{\mathcal{H}}_{p'}^c$ be such that $\|x\|_{\widetilde{\mathcal{H}}_{p'}^c} < 1$. Then there exists $(x_\sigma)^\bullet \in \mathcal{H}_p^c(\mathcal{U})$ such that $\|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} = \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{H_p^c(\sigma)} < 1$ and $x = \mathcal{E}_\mathcal{U}((x_\sigma)^\bullet) = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma$, where the weak-limit is in $L_p(\mathcal{M})$ if $1 < p < 2$ and in $L_2(\mathcal{M})$ if $2 \leq p < \infty$. Hence for $y \in \mathcal{M}$ we have $\tau(x^*y) = \lim_{\sigma,\mathcal{U}} \tau(x_\sigma^* y)$. Recall that for a fixed partition σ the Hölder inequality implies

$$|\tau(x_\sigma^* y)| \leq \|x_\sigma\|_{H_{p'}^c(\sigma)} \|y\|_{H_p^c(\sigma)}.$$

Taking the limit we get

$$|\tau(x^*y)| \leq \lim_{\sigma,\mathcal{U}} (\|x_\sigma\|_{H_{p'}^c(\sigma)} \|y\|_{H_p^c(\sigma)}) = (\lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{H_{p'}^c(\sigma)})(\lim_{\sigma,\mathcal{U}} \|y\|_{H_p^c(\sigma)}) \leq \|y\|_{\mathcal{H}_p^c}.$$

Since $\mathcal{M}$ is dense in $\mathcal{H}_p^c$, this shows that $x \in (\mathcal{H}_p^c)^*$ and

$$\|x\|_{(\mathcal{H}_p^c)^*} \leq \|x\|_{\widetilde{\mathcal{H}}_{p'}^c}.$$

Conversely, let $\varphi \in (\mathcal{H}_p^c)^*$ be a functional of norm less than one. Since $\mathcal{H}_p^c$ embeds isometrically into $\mathcal{H}_p^c(\mathcal{U})$ via the map $i_\mathcal{U}$, by the Hahn-Banach Theorem we can extend φ to a functional on $\mathcal{H}_p^c(\mathcal{U})$ of norm less than one. Then by Corollary 3.2.23 there exists $z = (z_\sigma)^\bullet \in \mathcal{H}_{p'}^c(\mathcal{U})$ of norm $\leq \sqrt{2}\gamma_p$ such that

$$\varphi(y) = (z | i_\mathcal{U}(y)), \quad \forall\, y \in \mathcal{H}_p^c.$$

Applying this to $y \in \mathcal{M}$ we get

$$\varphi(y) = (z|(y)^\bullet) = \lim_{\sigma,\mathcal{U}} \tau(z_\sigma^* y) = \tau(x^*y),$$

where $x = \mathcal{E}_\mathcal{U}(z)$ is in $\widetilde{\mathcal{H}}_{p'}^c$. By the density of $\mathcal{M}$ in $\mathcal{H}_p^c$ this proves that φ is represented by x and

$$\|x\|_{\widetilde{\mathcal{H}}_{p'}^c} \leq \|z\|_{\mathcal{H}_{p'}^c(\mathcal{U})} \leq \sqrt{2}\gamma_p \|x\|_{(\mathcal{H}_p^c)^*}.$$

□

The space $\widetilde{\mathcal{H}}_p^c$ actually coincides with the Hardy spaces defined previously.

Proposition 3.2.37. *Let $1 \leq p < \infty$. Then*

$$\mathcal{H}_p^c = \widetilde{\mathcal{H}}_p^c \quad \textit{with equivalent norms.}$$

We need the boundedness of the conditional expectation $\mathcal{E}_\mathcal{U}$ for $1 \leq p \leq 2$.

Proposition 3.2.38. *Let $1 \leq p \leq 2$. Then $\mathcal{E}_\mathcal{U} : \mathcal{H}_p^c(\mathcal{U}) \to \mathcal{H}_p^c$ is a bounded projection.*

Proof. By Lemma 3.2.24 it suffices to consider $x = (x_\sigma)^\bullet \in L_2(\mathcal{M}_\mathcal{U})$ such that $\|x\|_{\mathcal{H}_p^c(\mathcal{U})} < 1$. We can assume that for all σ, $\|x_\sigma\|_{H_p^c(\sigma)} < 1$. Then in this case $\mathcal{E}_\mathcal{U}(x)$ is the weak-limit of the x_σ's in L_2. Fix a partition σ'. By Lemma 3.2.12, for $\sigma' \subset \sigma$ we have

$$\|x_\sigma\|_{H_p^c(\sigma')} \leq \beta_p \|x_\sigma\|_{H_p^c(\sigma)} \leq \beta_p.$$

We see that $(x_\sigma)_{\sigma \supset \sigma'}$ is uniformly bounded in $H_p^c(\sigma')$. Moreover, for all $z \in H_{p'}^c(\sigma') \subset L_2(\mathcal{M})$, we have $\tau(z^*\mathcal{E}_\mathcal{U}(x)) = \lim_{\sigma,\mathcal{U}} \tau(z^* x_\sigma)$. Since $(H_p^c(\sigma'))^* = H_{p'}^c(\sigma')$, this means that $\mathcal{E}_\mathcal{U}(x)$ is the weak-limit of the x_σ's for $\sigma \supset \sigma'$ in $H_p^c(\sigma')$. We deduce that

$$\|\mathcal{E}_\mathcal{U}(x)\|_{H_p^c(\sigma')} \leq \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{H_p^c(\sigma')} \leq \beta_p.$$

Since this holds true for every partition σ', taking the limit we obtain

$$\|\mathcal{E}_\mathcal{U}(x)\|_{\mathcal{H}_p^c} \leq \beta_p \|x\|_{\mathcal{H}_p^c(\mathcal{U})}.$$

□

Proof of Proposition 3.2.37. We first consider the case $1 \leq p \leq 2$. Lemma 3.2.24 directly implies that $L_2(\mathcal{M})$ is dense in $\widetilde{\mathcal{H}}_p^c$. Hence it suffices to show that the norms $\|\cdot\|_{\mathcal{H}_p^c}$ and $\|\cdot\|_{\widetilde{\mathcal{H}}_p^c}$ are equivalent on $L_2(\mathcal{M})$. For $x \in L_2(\mathcal{M})$ we obviously have $x = \mathcal{E}_\mathcal{U}(i_\mathcal{U}(x)) = \mathcal{E}_\mathcal{U}((x)^\bullet)$, and

$$\|x\|_{\widetilde{\mathcal{H}}_p^c} = \inf_{x = \mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)} \|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} \leq \|(x)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} = \|x\|_{\mathcal{H}_p^c}.$$

Conversely, suppose that $\|x\|_{\widetilde{\mathcal{H}}_p^c} < 1$. Then there exists $(x_\sigma)^\bullet \in \mathcal{H}_p^c(\mathcal{U})$ such that $x = \mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$ and $\|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} < 1$. Proposition 3.2.38 implies that

$$\|x\|_{\mathcal{H}_p^c} = \|\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)\|_{\mathcal{H}_p^c} \leq \beta_p \|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} \leq \beta_p.$$

We now turn to the case $2 < p < \infty$. Let us first show that the norms $\|\cdot\|_{\mathcal{H}_p^c}$ and $\|\cdot\|_{\widetilde{\mathcal{H}}_p^c}$ are equivalent on $L_p(\mathcal{M})$. Let $x \in L_p(\mathcal{M})$. As previously, we can write $x = \mathcal{E}_\mathcal{U}((x)^\bullet)$ and obtain

$$\|x\|_{\widetilde{\mathcal{H}}_p^c} \leq \|x\|_{\mathcal{H}_p^c}.$$

Conversely, if $\|x\|_{\widetilde{\mathcal{H}}_p^c} < 1$ there exists $(x_\sigma)^\bullet \in \mathcal{H}_p^c(\mathcal{U})$ such that $x = \mathcal{E}_\mathcal{U}((x_\sigma)^\bullet) = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma$ and $\|(x_\sigma)^\bullet\|_{\mathcal{H}_p^c(\mathcal{U})} < 1$. We may assume that for all σ, $\|x_\sigma\|_{H_p^c(\sigma)} < 1$. The monotonicity Lemma 3.2.12 yields for each σ

$$\|x_\sigma\|_{\mathcal{H}_p^c} \leq \alpha_p \|x_\sigma\|_{H_p^c(\sigma)} \leq \alpha_p.$$

We see that $(x_\sigma)_\sigma$ is uniformly bounded in $\mathcal{H}_p^c$, which is reflexive by Lemma 3.2.14. Thus the weak-limit in $\mathcal{H}_p^c$ exists and is denoted by y. Since $L_2(\mathcal{M})$ is dense in $\widetilde{\mathcal{H}}_{p'}^c$, Proposition 3.2.36 implies that $\mathcal{H}_p^c$ embeds into $L_2(\mathcal{M})$. Then $y \in \mathcal{H}_p^c \subset L_2(\mathcal{M})$ and $x \in L_p(\mathcal{M}) \subset L_2(\mathcal{M})$ satisfy $\tau(z^*y) = \tau(z^*x)$ for all $z \in L_2(\mathcal{M})$. This means that $y = x$. We deduce that

$$\|x\|_{\mathcal{H}_p^c} \leq \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{\mathcal{H}_p^c} \leq \alpha_p.$$

It remains to see that $L_p(\mathcal{M})$ is dense in $\widetilde{\mathcal{H}}_p^c$. Since for $1 \leq p' < 2$, $\mathcal{H}_{p'}^c$ embeds into $L_{p'}(\mathcal{M})$ by Proposition 3.2.33 (ii), Proposition 3.2.36 implies that for $2 < p < \infty$, $L_p(\mathcal{M})$ is weak*-dense in $(\mathcal{H}_{p'}^c)^* = \widetilde{\mathcal{H}}_p^c$, so dense in $\widetilde{\mathcal{H}}_p^c$. This ends the proof. □

Finally, combining Proposition 3.2.36 with Proposition 3.2.37 we obtain the expected duality result.

Theorem 3.2.39. *Let $1 < p < \infty$. Then*

$$(\mathcal{H}_p^c)^* = \mathcal{H}_{p'}^c \quad \textit{with equivalent norms.}$$

Moreover,

$$k_p^{-1}\|x\|_{\mathcal{H}_{p'}^c} \leq \|x\|_{(\mathcal{H}_p^c)^*} \leq \|x\|_{\mathcal{H}_{p'}^c},$$

where $k_p = \sqrt{2}\gamma_p\alpha_{p'}$ for $1 < p < 2$ and $k_p = \sqrt{2}\gamma_p\beta_{p'}$ for $2 \leq p < \infty$.

Then Proposition 3.2.33 (iii) implies

Corollary 3.2.40. *Let $1 < p < 2$. Then $\mathcal{H}_p^c = \{x \in L_p(\mathcal{M}) \; : \; \|x\|_{\mathcal{H}_p^c} < \infty\}$.*

Remark 3.2.41. At the time of this writing we do not know if this result holds true for $p = 1$.

Note that Lemma 3.2.22 combined with Remark 3.2.35 (2) shows that $\widetilde{\mathcal{H}}_p^c$ is complemented in $K_p^c(\mathcal{U})$ for $1 < p < \infty$. Hence another consequence of Theorem 3.2.39 is

Corollary 3.2.42. *Let $1 < p < \infty$. Then $\mathcal{H}_p^c$ is complemented in $K_p^c(\mathcal{U})$.*

We can deduce from Corollary 3.2.18 (ii) the following interpolation result.

Corollary 3.2.43. *Let $1 < p_1, p_2 < \infty$ and $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$\mathcal{H}_p^c = [\mathcal{H}_{p_1}^c, \mathcal{H}_{p_2}^c]_\theta \qquad \textit{with equivalent norms.}$$

We will show later, in section 3.7, that this result still holds true for $p_1 = 1$.

3.2.6 Fefferman-Stein duality

In this subsection we establish the analogue of the Fefferman-Stein duality in the continuous setting. Our approach will be similar to that used in the previous subsection. Let us first introduce the ultraproduct space for $2 < p \leq \infty$

$$\widetilde{L_p^c\mathcal{MO}}(\mathcal{U}) = \prod_{\mathcal{U}} L_p^c MO(\sigma).$$

For $p = \infty$ we denote this space by $\widetilde{\mathcal{BMO}^c}(\mathcal{U})$. Then as in Remark 3.2.21 we can define the ultraproduct map of the componentwise inclusions $J'^c_p : \widetilde{L_p^c\mathcal{MO}}(\mathcal{U}) \to L_2(\widetilde{\mathcal{M}}_\mathcal{U})$ and compose by $\mathcal{E}_\mathcal{U}$, by taking the weak-limit in L_2. Then we get a bounded map

$$\mathcal{E}_\mathcal{U} \circ J'^c_p : \widetilde{L_p^c\mathcal{MO}}(\mathcal{U}) \to L_2(\mathcal{M}),$$

still denoted by $\mathcal{E}_\mathcal{U}$.

Definition 3.2.44. *(i) Let $2 < p < \infty$. We define the space $L_p^c\mathcal{MO}$ as the quotient of $\widetilde{L_p^c\mathcal{MO}}(\mathcal{U})$ by the kernel of the map $\mathcal{E}_\mathcal{U}$. The norm in $L_p^c\mathcal{MO}$ is given by the usual quotient norm*

$$\|x\|_{L_p^c\mathcal{MO}} = \inf_{x=w\text{-}\lim_\sigma x_\sigma} \|(x_\sigma)^\bullet\|_{\widetilde{L_p^c\mathcal{MO}}(\mathcal{U})} = \inf_{x=w\text{-}\lim_\sigma x_\sigma} \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{L_p^cMO(\sigma)}.$$

(ii) We define the space $\mathcal{BMO}^c$ as the space whose closed unit ball is given by the absolute convex set

$$B_{\mathcal{BMO}^c} = \overline{\{x = w\text{-}\lim_\sigma x_\sigma \text{ in } L_2 \; : \; \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{BMO^c(\sigma)} \le 1\}}^{\|\cdot\|_2}.$$

Then the norm in $\mathcal{BMO}^c$ is given by

$$\|x\|_{\mathcal{BMO}^c} = \inf\{C \ge 0 \; : \; x \in CB_{\mathcal{BMO}^c}\}.$$

For $2 < p < \infty$, the boundedness of $\mathcal{E}_\mathcal{U}$ on $\widetilde{L_p^c\mathcal{MO}}(\mathcal{U})$ immediately implies the completeness of the space $L_p^c\mathcal{MO}$. For $p = \infty$, note that we defined the $\mathcal{BMO}^c$-space in a slightly different way than for $2 < p < \infty$. We need the following general fact to prove that this defines a Banach space.

Lemma 3.2.45. *Let X be a Banach space and B be an absolutely convex subset of X satisfying*

(i) B is continuously embedded into the unit ball of X, i.e., there exists $D > 0$ such that

$$B \subset DB_X;$$

(ii) B is closed with respect to the norm $\|\cdot\|_X$.

Then the space Y whose unit ball is B, equipped with the norm

$$\|x\|_Y = \inf\{C \ge 0 \; : \; x \in CB\}$$

is a Banach space.

Proof. It is a well-known fact that $\|\cdot\|_Y$ defines a norm. Let $\sum_n x_{n\ge 1}$ be an absolutely converging series in $(Y, \|\cdot\|_Y)$. We may assume that $\|x_n\|_Y \le \frac{1}{2^n}$ for all $n \ge 1$. We want to show that this series converges in Y. We first remark that the series $\sum_n x_n$ is absolutely converging, and hence converging, in X. Then there exists $x \in X$ such that $x = \sum_n x_n$, where the convergence is with respect to $\|\cdot\|_X$. Thus

$$\sum_{n=1}^N x_n \overset{N\to\infty}{\longrightarrow} x \quad \text{in } X \quad \text{and} \quad \sum_{n=1}^N x_n \in B.$$

Indeed, we have

$$\Big\|\sum_{n=1}^N x_n\Big\|_Y \le \sum_{n=1}^N \|x_n\|_Y \le \sum_{n=1}^N \frac{1}{2^n} \le 1.$$

Using (ii), this shows that $x \in B$. It remains to see that the convergence also holds for the norm $\|\cdot\|_Y$. Let $\varepsilon > 0$. Let N_0 be such that $2^{N_0} \ge \varepsilon^{-1}$. We claim that for all $M > N \ge N_0$

$$y_{N,M} = \frac{1}{\varepsilon}\Big(\sum_{n=1}^M x_n - \sum_{n=1}^N x_n\Big) \in B.$$

Indeed, we have

$$\|y_{N,M}\|_Y = \frac{1}{\varepsilon}\Big\|\sum_{n=N+1}^{M} x_n\Big\|_Y \leq \frac{1}{\varepsilon}\sum_{n=N+1}^{M}\|x_n\|_Y \leq \frac{1}{\varepsilon}\sum_{n=N+1}^{M}\frac{1}{2^n} \leq \frac{1}{\varepsilon 2^N} \leq \frac{1}{\varepsilon 2^{N_0}} \leq 1.$$

Moreover, for $N \geq N_0$ fixed we have

$$y_{N,M} \overset{M\to\infty}{\longrightarrow} \frac{1}{\varepsilon}\Big(x - \sum_{n=1}^{N} x_n\Big) \quad \text{in } X.$$

Hence (ii) yields that $\frac{1}{\varepsilon}\Big(x - \sum\limits_{n=1}^{N} x_n\Big) \in B$, i.e.,

$$\Big\|x - \sum_{n=1}^{N} x_n\Big\|_Y \leq \varepsilon \quad \text{for all } \; N \geq N_0.$$

This proves that the series converges with respect to $\|\cdot\|_Y$ and ends the proof. $\square$

We can now prove that the space $\mathcal{BMO}^c$ defined previously is a Banach space. We apply Lemma 3.2.45 to $X = L_2(\mathcal{M})$ and $B = B_{\mathcal{BMO}^c}$. Then by the definition of $B_{\mathcal{BMO}^c}$, it is clear that the condition (ii) of the previous Lemma is satisfied. Moreover, since for $x \in L_2(\mathcal{M})$ and each σ we have $\|x\|_2 \leq \sqrt{2}\|x\|_{BMO^c(\sigma)}$, the condition (i) holds for $D = \sqrt{2}$. Hence the construction of the space $\mathcal{BMO}^c$ defines a Banach space.

Theorem 3.2.46. *Let $1 \leq p < 2$. Then*

$$(\mathcal{H}_p^c)^* = L_{p'}^c\mathcal{MO} \quad \textit{with equivalent norms.}$$

Moreover,

$$\lambda_p^{-1}\|x\|_{L_{p'}^c\mathcal{MO}} \leq \|x\|_{(\mathcal{H}_p^c)^*} \leq \sqrt{2}\|x\|_{L_{p'}^c\mathcal{MO}}.$$

Proof. For $1 < p < 2$, the proof is similar to that of Proposition 3.2.36. In this case we use the isometric embedding $i_{\mathcal{U}} : \mathcal{H}_p^c \to \widetilde{\mathcal{H}}_p^c(\mathcal{U})$, and by reflexivity of $\widetilde{\mathcal{H}}_p^c(\mathcal{U})$ we have

$$(\widetilde{\mathcal{H}}_p^c(\mathcal{U}))^* = \Big(\prod\nolimits_{\mathcal{U}} H_p^c(\sigma)\Big)^* = \prod\nolimits_{\mathcal{U}}(H_p^c(\sigma))^* = \prod\nolimits_{\mathcal{U}} L_{p'}^c MO(\sigma) = \widetilde{L_{p'}^c\mathcal{MO}}(\mathcal{U}),$$

where the constants in the equivalence of the norms come from the discrete case, i.e.,

$$\lambda_p^{-1}\|x\|_{\widetilde{L_{p'}^c\mathcal{MO}}(\mathcal{U})} \leq \|x\|_{(\widetilde{\mathcal{H}}_p^c(\mathcal{U}))^*} \leq \sqrt{2}\|x\|_{\widetilde{L_{p'}^c\mathcal{MO}}(\mathcal{U})}.$$

For $p = 1$, the inclusion $\mathcal{BMO}^c \subset (\mathcal{H}_1^c)^*$ follows easily from the discrete case and the density of $L_2(\mathcal{M})$ in $\mathcal{H}_1^c$. For the reverse inclusion we need Lemma 3.1.3. More precisely, let $\varphi \in (\mathcal{H}_1^c)^*$ be a functional of norm less than one. Since $\mathcal{H}_1^c$ embeds isometrically into $\widetilde{\mathcal{H}}_1^c(\mathcal{U})$ via the map $i_{\mathcal{U}}$, by the Hahn-Banach Theorem we can extend φ to a functional of norm less than one on $\widetilde{\mathcal{H}}_1^c(\mathcal{U})$. By Lemma 3.1.3 we see that the unit ball of $\prod_{\mathcal{U}}(H_1^c(\sigma))^* \cong \prod_{\mathcal{U}} BMO^c(\sigma)$ is weak*-dense in the unit ball of $(\prod_{\mathcal{U}} H_1^c(\sigma))^* = (\widetilde{\mathcal{H}}_1^c(\mathcal{U}))^*$. Then there exists a sequence $(z^\lambda)_\lambda$, where $z^\lambda = (z_\sigma^\lambda)^\bullet \in \widetilde{\mathcal{BMO}}^c(\mathcal{U})$ is of norm less than $\sqrt{3}$ for all λ, such that

$$\varphi(y) = \lim_{\lambda}(z^\lambda | i_{\mathcal{U}}(y)), \quad \forall\, y \in \mathcal{H}_1^c.$$

Applying this to $y \in L_2(\mathcal{M})$ we get

$$\varphi(y) = \lim_\lambda (z^\lambda|(y)^\bullet) = \lim_\lambda \lim_{\sigma,\mathcal{U}} \tau((z_\sigma^\lambda)^* y) = \lim_\lambda \tau((x^\lambda)^* y),$$

where $x^\lambda = \mathcal{E}_\mathcal{U}(z^\lambda) = w\text{-}\lim_{\sigma,\mathcal{U}} z_\sigma^\lambda$ is in $\mathcal{BMO}^c$ of norm less than $\sqrt{3}$. Since for all λ we have $\|z^\lambda\|_{\prod_\mathcal{U} L_2(\mathcal{M})} \leq \sqrt{2}\|z^\lambda\|_{\prod_\mathcal{U} BMO^c(\sigma)} \leq \sqrt{6}$, the sequence $(x^\lambda)_\lambda$ is uniformly bounded in $L_2(\mathcal{M})$. Setting $x = w\text{-}\lim_\lambda x^\lambda$ in L_2 we obtain $\varphi(y) = \tau(x^* y)$ for all $y \in L_2(\mathcal{M})$. We can approximate the weak-limit x by convex combinations of the x^λ's in the L_2-norm. Since $x^\lambda \in \sqrt{3}B_{\mathcal{BMO}^c}$ for all λ, the convexity of the unit ball of $\mathcal{BMO}^c$ implies that any convex combination $\sum_m \alpha_m x^{\lambda_m}$ is still in $\sqrt{3}B_{\mathcal{BMO}^c}$. Thus by the definition of $B_{\mathcal{BMO}^c}$, we obtain that $x \in \sqrt{3}B_{\mathcal{BMO}^c}$. By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_1^c$ this proves that φ is represented by x and

$$\|x\|_{\mathcal{BMO}^c} \leq \sqrt{3}\|x\|_{(\mathcal{H}_1^c)^*}.$$

□

This duality implies the following result.

Corollary 3.2.47. *Let $2 < p \leq \infty$. Let $(x_\lambda)_\lambda$ be a sequence in $L_2(\mathcal{M})$ such that $\|x_\lambda\|_{L_p^c\mathcal{MO}} \leq 1$ for all λ and $x = w\text{-}\lim_\lambda x_\lambda$ in L_2. Then $x \in L_p^c\mathcal{MO}$ with $\|x\|_{L_p^c\mathcal{MO}} \leq \sqrt{2}\lambda_p$.*

Proof. Using Theorem 3.2.46 and the density of $L_2(\mathcal{M})$ in $\mathcal{H}_{p'}^c$, we can write

$$\|x\|_{L_p^c\mathcal{MO}} \leq \lambda_p \sup_{y \in L_2(\mathcal{M}), \|y\|_{\mathcal{H}_{p'}^c} \leq 1} |\tau(x^* y)|.$$

Note that for all $y \in L_2(\mathcal{M}), \|y\|_{\mathcal{H}_{p'}^c} \leq 1$ we have

$$|\tau(x^* y)| \leq \limsup_\lambda |\tau(x_\lambda^* y)| \leq \sqrt{2} \limsup_\lambda \|x_\lambda\|_{L_p^c\mathcal{MO}} \|y\|_{\mathcal{H}_{p'}^c} \leq \sqrt{2}.$$

Thus $x \in L_p^c\mathcal{MO}$ with $\|x\|_{L_p^c\mathcal{MO}} \leq \sqrt{2}\lambda_p$. □

Combining Theorem 3.2.46 and Theorem 3.2.39 we immediately get the

Corollary 3.2.48. *Let $2 < p < \infty$. Then*

$$L_p^c\mathcal{MO} = \mathcal{H}_p^c \quad \textit{with equivalent norms.}$$

Remark 3.2.49. In particular, we deduce the following properties for $L_p^c\mathcal{MO}$, $2 < p < \infty$:

(i) $L_p^c\mathcal{MO}$ is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.

(ii) $L_p(\mathcal{M})$ is norm dense in $L_p^c\mathcal{MO}$.

(iii) For $x \in L_p(\mathcal{M})$,

$$\|x\|_{L_p^c\mathcal{MO}} = \lim_{q \to p} \|x\|_{L_q^c\mathcal{MO}} \simeq \lim_{\sigma,\mathcal{U}} \|x\|_{L_p^c MO(\sigma)}$$

for every ultrafilter $\mathcal{U}$. In particular, up to equivalent norms, $L_p^c\mathcal{MO}$ is the completion of $L_p(\mathcal{M})$ with respect to the norm $\lim_{\sigma,\mathcal{U}} \|\cdot\|_{L_p^c MO(\sigma)}$.

(iv) The $\|\cdot\|_{L_p^c MO(\sigma)}$-norm is decreasing in σ (up to a constant).

Note that for (iii), if $x \in L_p(\mathcal{M})$ the fact that $\|x\|_{L_p^c\mathcal{MO}} \simeq \|x\|_{\mathcal{H}_p^c}$ combined with Lemma 3.2.25 ensures that $\lim_{q\to p} \|x\|_{L_q^c\mathcal{MO}}$ exists. Since for $q < p < r$ we have $\|x\|_{L_q^c\mathcal{MO}} \leq \|x\|_{L_p^c\mathcal{MO}} \leq \|x\|_{L_r^c\mathcal{MO}}$, sending q and r to p we obtain that the limit is in fact equal to $\|x\|_{L_p^c\mathcal{MO}}$.

Concerning the case $p = \infty$, we can also deduce some nice properties for $\mathcal{BMO}^c$ from Theorem 3.2.46.

Corollary 3.2.50. *(i) $\mathcal{BMO}^c$ is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.*

(ii) $\mathcal{M}$ is weak-dense in $\mathcal{BMO}^c$.*

(iii) For $x \in \mathcal{M}$,

$$\|x\|_{\mathcal{BMO}^c} \simeq \sup_{2<p<\infty} \|x\|_{L_p^c\mathcal{MO}} \leq \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}$$

for every ultrafilter $\mathcal{U}$.

(iv) The $\|\cdot\|_{BMO^c(\sigma)}$-norm is decreasing in σ (up to a constant). More precisely, for $x \in \mathcal{M}$ and $\sigma \subset \sigma'$ we have

$$\|x\|_{BMO^c(\sigma')} \leq 2\|x\|_{BMO^c(\sigma)}.$$

Proof. (i) and (ii) follow directly from Theorem 3.2.46 and Theorem 3.2.13, Proposition 3.2.33 respectively. For $x \in \mathcal{M}$ and $2 < p < \infty$, it is trivial that $\|x\|_{L_p^c\mathcal{MO}} \leq \|x\|_{\mathcal{BMO}^c}$. Conversely, by the density of $\mathcal{M}$ in $\mathcal{H}_1^c$ we have

$$\|x\|_{\mathcal{BMO}^c} \leq \sqrt{3}\|x\|_{(\mathcal{H}_1^c)^*} = \sqrt{3} \sup_{y\in\mathcal{M},\|y\|_{\mathcal{H}_1^c}\leq 1} |\tau(x^*y)|.$$

Let $\varepsilon > 0$. By Lemma 3.2.25, for each $y \in \mathcal{M}$, $\|y\|_{\mathcal{H}_1^c} \leq 1$ there exists $p(y) > 1$ such that $\|y\|_{\mathcal{H}_{p(y)}^c} \leq 1 + \varepsilon$. Applying Theorem 3.2.46 to $\frac{1}{p(y)} + \frac{1}{p(y)'} = 1$ we get

$$|\tau(x^*y)| \leq \sqrt{2}\|x\|_{L_{p(y)'}^c\mathcal{MO}}\|y\|_{\mathcal{H}_{p(y)}^c} \leq \sqrt{2}(1+\varepsilon) \sup_{2<p<\infty} \|x\|_{L_p^c\mathcal{MO}}.$$

Sending ε to 0, we obtain

$$\sup_p \|x\|_{L_p^c\mathcal{MO}} \leq \|x\|_{\mathcal{BMO}^c} \leq \sqrt{6} \sup_{2<p<\infty} \|x\|_{L_p^c\mathcal{MO}}.$$

Then by Remark 3.2.49 we deduce

$$\|x\|_{\mathcal{BMO}^c} \simeq \sup_p \|x\|_{L_p^c\mathcal{MO}} \simeq \sup_{2<p<\infty} \lim_{\sigma,\mathcal{U}} \|x\|_{L_p^cMO(\sigma)} \leq \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}.$$

Finally, (iv) comes from the reversed monotonicity result for the $H_1^c(\sigma)$-norms by duality. But this approach yields a constant $\sqrt{12}$, which can be improved by a direct proof that we include below. Let $x \in \mathcal{M}$ and $\sigma \subset \sigma'$. Fix $u \in \sigma'$, there exists a unique element $s(u) \in \sigma$, satisfying $s(u)^- \leq u^- < u \leq s(u)$. Observe that for $b \in \mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}$ we have by contractivity of the conditional expectation

$$\|\mathcal{E}_u|b - \mathcal{E}_{u^-}(b)|^2\|_\infty \leq 2(\|\mathcal{E}_u|b|^2\|_\infty + \|\mathcal{E}_u|\mathcal{E}_{u^-}(b)|^2\|_\infty \leq 4\|\mathcal{E}_u(\mathcal{E}_{s(u)}|b|^2)\|_\infty \leq 4\|\mathcal{E}_{s(u)}|b|^2\|_\infty.$$

Applying this to

$$b = \sum_{s\in\sigma, s\geq s(u)} e_{s,0} \otimes d_s^\sigma(x) \in \mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}$$

we get

$$\|\mathcal{E}_u|b - \mathcal{E}_{u^-}(b)|^2\|_{\mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}} = \Big\|\mathcal{E}_u\Big(\sum_{v\in\sigma',v\geq u}|d_v^{\sigma'}(b)|^2\Big)\Big\|_{\mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}}$$
$$\leq 4\|\mathcal{E}_{s(u)}|b|^2\|_{\mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}} = 4\Big\|\mathcal{E}_{s(u)}\Big(\sum_{s\in\sigma,s\geq s(u)}|d_s^{\sigma}(x)|^2\Big)\Big\|_\infty.$$

Recall that

$$d_v^{\sigma'}(d_s^\sigma(x)) = \begin{cases} d_v^{\sigma'}(x) & \text{if} \quad s^- \leq v^- < v \leq s \\ 0 & \text{otherwise} \end{cases}.$$

Note that for $v \in \sigma', v \geq u$ fixed there exists a unique element $s(v) \in \sigma$ satisfying $s(v)^- \leq v^- < v \leq s(v)$. Moreover, $s(\cdot)$ is monotonous, i.e., $v \geq u$ implies $s(v) \geq s(u)$. Hence

$$d_v^{\sigma'}(b) = \sum_{s\in\sigma,s\geq s(u)} e_{s,0}\otimes d_v^{\sigma'}(d_s^\sigma(x)) = e_{s(v),0}\otimes d_v^{\sigma'}(x),$$

and

$$\Big\|\mathcal{E}_u\Big(\sum_{v\in\sigma',v\geq u}|d_v^{\sigma'}(b)|^2\Big)\Big\|_{\mathcal{B}(\ell_2(\sigma))\overline{\otimes}\mathcal{M}} = \Big\|\mathcal{E}_u\Big(\sum_{v\in\sigma',v\geq u}|d_v^{\sigma'}(x)|^2\Big)\Big\|_\infty.$$

At the end we showed that for each $u \in \sigma'$,

$$\|\mathcal{E}_u|x - \mathcal{E}_{u^-}(x)|^2\|_\infty^{1/2} \leq 2\|x\|_{BMO^c(\sigma)},$$

which yields the required result by taking the supremum over $u \in \sigma'$. □

Remark 3.2.51. At the time of this writing, we do not know if for $x \in \mathcal{M}$ we have

$$\|x\|_{\mathcal{BMO}^c} \simeq \lim_{\sigma,\mathcal{U}} \|x\|_{BMO^c(\sigma)}.$$

We end this subsection with the following characterization of the $L_p^c\mathcal{MO}$-spaces. Observe that this characterization also holds true for $p = \infty$, hence this allows us to consider the spaces $L_p^c\mathcal{MO}$ and $\mathcal{BMO}^c$ in a similar way.

Proposition 3.2.52. *Let $2 < p \leq \infty$. Then the unit ball of $L_p^c\mathcal{MO}$ is equivalent to*

$$\mathcal{B}_p = \{x \in L_2(\mathcal{M}) \ : \ x = L_2\text{-}\lim_\lambda x_\lambda, \ \lim_{\sigma,\mathcal{U}} \|x_\lambda\|_{L_p^cMO(\sigma)} \leq 1, \ \forall\ \lambda\}.$$

Proof. For $p = \infty$, it is obvious that $\mathcal{B}_\infty \subset B_{\mathcal{BMO}^c}$. For $2 < p < \infty$, Corollary 3.2.47 implies that $\mathcal{B}_p \subset \sqrt{2}\lambda_p B_{L_p^c\mathcal{MO}}$. Conversely, let $x \in B_{L_p^c\mathcal{MO}}$. It suffices to consider $x = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma$ in L_2 with $\|x_\sigma\|_{L_p^cMO(\sigma)} \leq 1$. For a fixed partition σ', since the $L_p^cMO(\sigma)$-norms are decreasing we have

$$\lim_{\sigma,\mathcal{U}} \|x_{\sigma'}\|_{L_p^cMO(\sigma)} \leq k_p\|x_{\sigma'}\|_{L_p^cMO(\sigma')} \leq k_p.$$

Moreover, the family $(x_\sigma)_\sigma$ is uniformly bounded in $L_2(\mathcal{M})$. Then x is the limit in L_2-norm of convex combinations of the x_σ's. Let $y = \sum_m \alpha_m x_{\sigma^m}$ be such a convex combination, then

$$\lim_{\sigma,\mathcal{U}} \|y\|_{L_p^cMO(\sigma)} \leq \sum_m \alpha_m \lim_{\sigma,\mathcal{U}} \|x_{\sigma^m}\|_{L_p^cMO(\sigma)} \leq k_p.$$

Hence $x \in k_p\mathcal{B}_p$. □

3.2.7 The Hardy space $\mathcal{H}_p$

The whole theory developed previously for the column spaces still holds true for the row spaces. Indeed, by considering the adjoint we get the analoguous results for $\mathcal{H}_p^r$. In this subsection we discuss the combination of the column and row Hardy spaces. We start with the analogue of Theorem 3.2.7 in our setting. Then we discuss briefly the space $\mathcal{H}_1$, and give another characterization for this space.

Burkholder-Gundy inequalities

Our aim is to establish the analogue of the noncommutative Burkholder-Gundy inequalities in the continuous setting. The approach we will use is philosophically close to arguments from nonstandard analysis. Indeed, we will derive the result for continuous filtrations as transfer from finite partitions. More precisely, we will first prove the inequalities for the ultraproduct spaces and then will transfer it to continuous filtrations by applying the conditional expectation $\mathcal{E}_{\mathcal{U}}$, which coincides with the weak-limit (this corresponds to the standard part operation in nonstandard analysis). Let us first give the natural definition for the Hardy space $\mathcal{H}_p$.

Definition 3.2.53. *Let $1 \leq p < \infty$. We define*

$$\mathcal{H}_p = \begin{cases} \mathcal{H}_p^c + \mathcal{H}_p^r & \textit{for} \quad 1 \leq p < 2 \\ \mathcal{H}_p^c \cap \mathcal{H}_p^r & \textit{for} \quad 2 \leq p < \infty \end{cases},$$

where the sum is taken in $L_p(\mathcal{M})$ and the intersection in $L_2(\mathcal{M})$.

Observe that for $2 \leq p < \infty$, by applying the noncommutative Burkholder-Gundy inequalities in the discrete case for each partition σ and taking the limit in σ we immediately obtain

$$\|x\|_p \simeq \max(\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}) \quad \text{for } x \in L_p(\mathcal{M}).$$

This means that

$$L_p(\mathcal{M}) = \overline{L_p(\mathcal{M})}^{\|\cdot\|_{\mathcal{H}_p^c \cap \mathcal{H}_p^r}} \quad \text{for } 2 \leq p < \infty.$$

However this result is too weak, we would like to prove that $L_p(\mathcal{M}) = \mathcal{H}_p^c \cap \mathcal{H}_p^r$ for $2 \leq p < \infty$. To obtain this stronger result, we use a dual approach and first consider the case $1 < p < 2$. The Burkholder-Gundy Theorem 3.2.7 for $1 < p < 2$ in the discrete case applied to each partition σ immediately implies the analoguous result for the ultraproduct spaces introduced in subsection 3.2.3. Recall that for $1 < p < 2$ we defined $J_p^c : \widetilde{\mathcal{H}}_p^c(\mathcal{U}) \to L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})$, the (non necessarily injective) ultraproduct map of the componentwise inclusions. We can similarly consider $J_p^r : \widetilde{\mathcal{H}}_p^r(\mathcal{U}) \to L_p(\widetilde{\mathcal{M}}_{\mathcal{U}})$. For the sake of simplicity we will denote these two maps by the same notation J_p in the sequel.

Proposition 3.2.54. *Let $1 < p < 2$. Then*

$$L_p(\widetilde{\mathcal{M}}_{\mathcal{U}}) = J_p(\widetilde{\mathcal{H}}_p^c(\mathcal{U})) + J_p(\widetilde{\mathcal{H}}_p^r(\mathcal{U})) \quad \textit{with equivalent norms.}$$

Moreover,

$$\alpha_p^{-1}\|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U}))+\widetilde{\mathcal{H}}_p^r(\mathcal{U})} \leq \|J_p(x)\|_p \leq \beta_p\|x\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U}))+\widetilde{\mathcal{H}}_p^r(\mathcal{U})}.$$

By the characterization of $L_p(\mathcal{M}_{\mathcal{U}})$ given in Lemma 3.1.7, we see that the map J_p preserves the regularized spaces by mapping $\mathcal{H}_p^c(\mathcal{U})$ and $\mathcal{H}_p^r(\mathcal{U})$ to $L_p(\mathcal{M}_{\mathcal{U}})$. We can now state the analoguous result for the regularized spaces.

Proposition 3.2.55. *Let $1 < p < 2$. Then*

$$L_p(\mathcal{M}_\mathcal{U}) = J_p(\mathcal{H}_p^c(\mathcal{U})) + J_p(\mathcal{H}_p^r(\mathcal{U})) \quad \textit{with equivalent norms.}$$

Proof. The inclusion $J_p(\mathcal{H}_p^c(\mathcal{U})) + J_p(\mathcal{H}_p^r(\mathcal{U})) \subset L_p(\mathcal{M}_\mathcal{U})$ is trivial. Conversely, let $x \in L_p(\mathcal{M}_\mathcal{U})$ be such that $\|x\|_p < 1$. Then by Lemma 3.1.7 we can find a sequence $(x_n)_{n\geq 1}$ such that for each n, there exists $p_n > p$ satisfying

(i) $x_n \in L_{p_n}(\widetilde{\mathcal{M}}_\mathcal{U})$ and $\|I_{p_n,p}(x_n)\|_p < \frac{1}{2^n}$;

(ii) $x = \sum_{n\geq 1} I_{p_n,p}(x_n)$ in $L_p(\widetilde{\mathcal{M}}_\mathcal{U})$.

We may assume in addition that

(iii) for all $n \geq 1$, $\|x_n\|_{p_n} < \frac{1}{2^n}$.

Indeed, since $I_{p_n,p}(x_n) \in L_p(\mathcal{M}_\mathcal{U})$ and $\mathcal{M}_\mathcal{U}$ is finite, we have

$$\|I_{p_n,p}(x_n)\|_p = \lim_{q \overset{>}{\to} p} \|I_{p_n,q}(x_n)\|_q < \frac{1}{2^n}.$$

Thus there exists $q_n > p$ such that $\|I_{p_n,q_n}(x_n)\|_{q_n} < \frac{1}{2^n}$. If $p_n \leq q_n$ then we get the required estimate, otherwise we replace p_n by q_n and x_n by $I_{p_n,q_n}(x_n)$ to obtain the assumption. We can also suppose that p_n is not too large, say $p < p_n < 2p$. We now apply Proposition 3.2.54 to each x_n and p_n. For all $n \geq 1$, there exist $x_n^c \in \widetilde{\mathcal{H}}_{p_n}^c(\mathcal{U})$ and $x_n^r \in \widetilde{\mathcal{H}}_{p_n}^r(\mathcal{U})$ such that

$$x_n = J_{p_n}(x_n^c) + J_{p_n}(x_n^r)$$

and

$$\|x_n^c\|_{\widetilde{\mathcal{H}}_{p_n}^c(\mathcal{U})} + \|x_n^r\|_{\widetilde{\mathcal{H}}_{p_n}^r(\mathcal{U})} \leq \alpha_{p_n} \|x_n\|_{p_n} < \alpha_{p_n} \frac{1}{2^n},$$

where the last inequality comes from assumption (iii). Then $\|I_{p_n,p}(x_n^c)\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} \leq \alpha_{p_n} \frac{1}{2^n}$ and the series $\sum_{n\geq 1} I_{p_n,p}(x_n^c)$ converges in $\widetilde{\mathcal{H}}_p^c(\mathcal{U})$. We have the similar convergence for x_n^r and we set

$$x^c = \sum_{n\geq 1} I_{p_n,p}(x_n^c) \in \mathcal{H}_p^c(\mathcal{U}) \quad \text{and} \quad x^r = \sum_{n\geq 1} I_{p_n,p}(x_n^r) \in \mathcal{H}_p^r(\mathcal{U}).$$

On the one hand we have

$$\sum_{n=1}^{N} I_{p_n,p}(x_n) \overset{N\to\infty}{\longrightarrow} x \quad \text{in} \quad L_p(\widetilde{\mathcal{M}}_\mathcal{U}).$$

On the other hand, since $I_{p_n,p} \circ J_{p_n} = J_p \circ I_{p_n,p}$ and by the continuity of J_p, we can write for $N \geq 1$

$$\begin{aligned}
\sum_{n=1}^{N} I_{p_n,p}(x_n) &= \sum_{n=1}^{N} I_{p_n,p}(J_{p_n}(x_n^c)) + \sum_{n=1}^{N} I_{p_n,p}(J_{p_n}(x_n^r)) \\
&= J_p\Big(\sum_{n=1}^{N} I_{p_n,p}(x_n^c)\Big) + J_p\Big(\sum_{n=1}^{N} I_{p_n,p}(x_n^r)\Big) \\
&\overset{N\to\infty}{\longrightarrow} J_p(x^c) + J_p(x^r) \quad \text{in} \quad L_p(\widetilde{\mathcal{M}}_\mathcal{U}).
\end{aligned}$$

Finally, by the uniqueness of the limit we obtain

$$x = J_p(x^c) + J_p(x^r)$$

with

$$\|x^c\|_{\mathcal{H}_p^c(\mathcal{U})} + \|x^r\|_{\mathcal{H}_p^r(\mathcal{U})} \leq \sup_{p<q<2p} \alpha_q.$$

□

We can now establish the noncommutative Burkholder-Gundy inequalities in the continuous setting.

Theorem 3.2.56. *Let $1 < p < \infty$. Then*

$$L_p(\mathcal{M}) = \mathcal{H}_p \quad \textit{with equivalent norms.}$$

Proof. For $1 < p < 2$, we apply $\mathcal{E}_\mathcal{U}$ to Proposition 3.2.55. Since $\mathcal{E}_\mathcal{U}$ is bounded on $\mathcal{H}_p^c(\mathcal{U})$ by Proposition 3.2.38, it suffices to observe that the following diagram is commuting

$$\begin{tikzcd} \mathcal{H}_p^c(\mathcal{U}) \arrow[r, "J_p^c"] \arrow[d, "\mathcal{E}_\mathcal{U}"] & L_p(\mathcal{M}_\mathcal{U}) \arrow[d, "\mathcal{E}_\mathcal{U}"] \\ \mathcal{H}_p^c \arrow[r, "id"'] & L_p(\mathcal{M}) \end{tikzcd}$$

This diagram is obviously commuting on L_2, hence everywhere by the density of L_2 in the considered spaces.

The case $2 < p' < \infty$ follows by duality. Indeed, by the density of $L_2(\mathcal{M})$ in both spaces $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$ for $1 < p < 2$, the intersection $\mathcal{H}_p^c \cap \mathcal{H}_p^r$ is dense in $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$. Then the dual space of the sum is the intersection of the dual spaces, and Theorem 3.2.39 implies that

$$(\mathcal{H}_p^c + \mathcal{H}_p^r)^* = (\mathcal{H}_p^c)^* \cap (\mathcal{H}_p^r)^* = \mathcal{H}_{p'}^c \cap \mathcal{H}_{p'}^r$$

with equivalent norms. Finally, the duality $(L_p(\mathcal{M}))^* = L_{p'}(\mathcal{M})$ and the first part of the proof yield the Burkholder-Gundy inequalities for $2 < p' < \infty$. Since the case $p = 2$ is trivial, this concludes the proof. □

Fefferman-Stein duality for $\mathcal{H}_p$

Let us describe the dual space of $\mathcal{H}_p$ for $1 \leq p < 2$ as follows.

Definition 3.2.57. *Let $2 < p \leq \infty$. We define*

$$L_p\mathcal{MO} = L_p^c\mathcal{MO} \cap L_p^r\mathcal{MO},$$

where the intersection is taken in $L_2(\mathcal{M})$ and the norm is given by the usual intersection norm

$$\|x\|_{L_p\mathcal{MO}} = \max\big(\|x\|_{L_p^c\mathcal{MO}}\,,\ \|x\|_{L_p^r\mathcal{MO}}\big).$$

For $p = \infty$ we use $\mathcal{BMO}$ instead of $L_\infty\mathcal{MO}$.

Remark 3.2.58. Observe that by Remark 3.2.49 (iii) we have for $x \in L_p(\mathcal{M})$

$$\|x\|_{L_p\mathcal{MO}} \simeq \lim_{q\to p} \|x\|_{L_q\mathcal{MO}} \simeq \lim_{\sigma,\mathcal{U}} \|x\|_{L_pMO(\sigma)} \quad \text{for } 2 < p < \infty.$$

Since $L_2(\mathcal{M})$ is dense in $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$ for $1 \leq p < 2$, we see that the intersection $\mathcal{H}_p^c \cap \mathcal{H}_p^r$ is also dense in $\mathcal{H}_p^c$ and in $\mathcal{H}_p^r$. Hence by definition of $\mathcal{H}_p$ and Theorem 3.2.46 we immediately get the following duality.

Theorem 3.2.59. *Let $1 \leq p < 2$. Then*

$$(\mathcal{H}_p)^* = L_{p'}\mathcal{MO} \quad \textit{with equivalent norms.}$$

Another characterization of $\mathcal{H}_1$

We end this subsection with a discussion on the space $\mathcal{H}_1$. For $x \in L_2(\mathcal{M})$ we consider

$$\|x\|_{\check{\mathcal{H}}_1} = \lim_{p\to 1} \|x\|_{\mathcal{H}_p}.$$

Then the inequalities

$$\beta_1^{-1}\|x\|_1 \leq \|x\|_{\check{\mathcal{H}}_1} \leq \|x\|_2$$

ensure that this defines a norm on $L_2(\mathcal{M})$. Since the $\mathcal{H}_p$-norm is decreasing in p, the limit is in fact an infimum, which exists for $\|x\|_{\mathcal{H}_p}$ is then a decreasing sequence bounded by below.

Definition 3.2.60. *We define the space $\check{\mathcal{H}}_1$ as the completion of $L_2(\mathcal{M})$ with respect to the norm $\|\cdot\|_{\check{\mathcal{H}}_1}$.*

It is clear that $\|x\|_{\mathcal{H}_1} \leq \|x\|_{\check{\mathcal{H}}_1}$ for $x \in L_2(\mathcal{M})$. We show that these two norms are actually equivalent by using a dual approach. We fix an ultrafilter $\mathcal{V}$ on $[1,\infty)$ containing the filter base $\{]1, 1+\frac{1}{n}] \ : \ n \geq 1\}$. Note that if $(a_p)_{p>1}$ is convergent as $p \to 1$, then $\lim_{p\to 1} a_p = \lim_{p,\mathcal{V}} a_p$. We will need the following fact, which is a direct consequence of Lemma 3.2.25.

Lemma 3.2.61. *Let $(x_p)_{p\geq 1}$ be a uniformly bounded family in $L_2(\mathcal{M})$ and set $x = w\text{-}\lim_{p,\mathcal{V}} x_p$ in L_2. Then*

$$\|x\|_{(\mathcal{H}_1^c)^*} \leq \lim_{p,\mathcal{V}} \|x\|_{(\mathcal{H}_p^c)^*}.$$

Proof. By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_1^c$ we can write

$$\|x\|_{(\mathcal{H}_1^c)^*} = \sup_{y\in L_2(\mathcal{M}), \|y\|_{\mathcal{H}_1^c}\leq 1} |\tau(x^*y)| = \sup_{y\in L_2(\mathcal{M}), \|y\|_{\mathcal{H}_1^c}\leq 1} |\lim_{p,\mathcal{V}} \tau(x_p^*y)|.$$

For $\varepsilon > 0$ and $y \in L_2(\mathcal{M}), \|y\|_{\mathcal{H}_1^c} \leq 1$, by Lemma 3.2.25 there exists $p(y) > 1$ such that $\|y\|_{\mathcal{H}_{p(y)}^c} \leq 1+\varepsilon$. Then for $1 < p \leq p(y)$, Theorem 3.2.39 implies

$$|\tau(x_p^*y)| \leq \|x_p\|_{(\mathcal{H}_{p(y)}^c)^*}\|y\|_{\mathcal{H}_{p(y)}^c} \leq \|x_p\|_{(\mathcal{H}_p^c)^*}(1+\varepsilon).$$

Sending $\varepsilon \to 0$ and taking the limit in p over $\mathcal{V}$ we get the result. $\square$

We can now prove that this new Hardy space $\check{\mathcal{H}}_1$ coincides with $\mathcal{H}_1$.

Theorem 3.2.62. *We have*

$$\mathcal{H}_1 = \check{\mathcal{H}}_1 \quad \textit{isometrically.}$$

Proof. We will prove that $(\check{\mathcal{H}}_1)^* \subset (\mathcal{H}_1)^*$ contractively. Then we will get the inequality $\|x\|_{\check{\mathcal{H}}_1} \leq \|x\|_{\mathcal{H}_1}$. Since the reverse inequality is trivial on $L_2(\mathcal{M})$, we will obtain that $\|x\|_{\check{\mathcal{H}}_1} = \|x\|_{\mathcal{H}_1}$ for $x \in L_2(\mathcal{M})$. By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_1$ and $\check{\mathcal{H}}_1$, we will get the required result. Let $\varphi \in (\check{\mathcal{H}}_1)^*$ be a functional of norm less than one. Observe that we have an isometric embedding

$$i_{\mathcal{V}} : \check{\mathcal{H}}_1 \to \prod_{\mathcal{V}} \mathcal{H}_p$$

defined by $i_{\mathcal{V}}(x) = (x)^\bullet$ for $x \in L_2(\mathcal{M})$. Since by Lemma 3.1.3, the unit ball of $\prod_{\mathcal{V}}(\mathcal{H}_p)^*$ is weak*-dense in the unit ball of $\Big(\prod_{\mathcal{V}} \mathcal{H}_p\Big)^*$, there exists a sequence $z^\lambda = (z_p^\lambda)^\bullet \in \prod_{\mathcal{V}}(\mathcal{H}_p)^*$ such that

$$\lim_{p,\mathcal{V}} \|z_p^\lambda\|_{(\mathcal{H}_p)^*} \leq 1 \quad \text{for all } \lambda \quad \text{and} \quad \varphi(y) = \lim_{\lambda} (z^\lambda | i_{\mathcal{V}}(y)), \quad \forall\, y \in \check{\mathcal{H}}_1.$$

Applying this to $y \in L_2(\mathcal{M})$ we get

$$\varphi(y) = \lim_{\lambda} \lim_{p,\mathcal{V}} \tau((z_p^\lambda)^* y) = \lim_{\lambda} \tau((z^\lambda)^* y),$$

where $z^\lambda = w\text{-}\lim_{p,\mathcal{V}} z_p^\lambda$ in $L_2(\mathcal{M})$. Note that this weak-limit exists for the family $(z_p^\lambda)_p$ is uniformly bounded in L_2. Finally, since the family $(z^\lambda)_\lambda$ is also uniformly bounded in L_2, we set $z = w\text{-}\lim_\lambda z^\lambda$ in $L_2(\mathcal{M})$. Then

$$\varphi(y) = \tau(z^* y), \quad \forall\, y \in L_2(\mathcal{M}).$$

Since $L_2(\mathcal{M})$ is dense in $\check{\mathcal{H}}_1$, φ is represented by z and it remains to show that $z \in (\mathcal{H}_1)^* = (\mathcal{H}_1^c + \mathcal{H}_1^r)^* = (\mathcal{H}_1^c)^* \cap (\mathcal{H}_1^r)^*$. By the density of $L_2(\mathcal{M})$ in $\mathcal{H}_1^c$, it suffices to show that $z^\lambda \in (\mathcal{H}_1^c)^*$ with relevant constant independent of λ. The row estimate is similar. Lemma 3.2.61 yields

$$\|z^\lambda\|_{(\mathcal{H}_1^c)^*} \leq \lim_{p,\mathcal{V}} \|z_p^\lambda\|_{(\mathcal{H}_p^c)^*} \leq 1,$$

and this ends the proof. $\square$

An immediate consequence of Theorem 3.2.62 is that the space $\check{\mathcal{H}}_1$ embeds into $L_1(\mathcal{M})$. This characterization will be useful for some approximation arguments in the sequel.

3.3 The h_p^c-spaces

In this section we consider the conditioned version of Hardy spaces, and study their continuous analogue. We will follow the same approach as for the $\mathcal{H}_p^c$-spaces in the previous section. Since the theory of h_p^c-spaces is similar to that of $\mathcal{H}_p^c$-spaces, we will not detail all proofs and will emphasize on the main differences.

3.3.1 The discrete case

As in section 3.2, we start by recalling the definitions of the conditioned Hardy spaces of noncommutative martingales in the discrete case and some well-known results. Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration. Following [24], we introduce the column and row conditioned square functions relative to a (finite) martingale $x = (x_n)_{n\geq 0}$ in $L_\infty(\mathcal{M})$:

$$s_c(x) = \Big(\sum_{n=0}^{\infty} \mathcal{E}_{n-1}|d_n(x)|^2\Big)^{1/2} \quad \text{and} \quad s_r(x) = \Big(\sum_{n=0}^{\infty} \mathcal{E}_{n-1}|d_n(x)^*|^2\Big)^{1/2},$$

where by convention we set $\mathcal{E}_{-1} = \mathcal{E}_0$. For $1 \leq p < \infty$ we define h_p^c (resp. h_p^r) as the completion of all finite L_∞-martingales under the norm $\|x\|_{h_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{h_p^r} = \|s_r(x)\|_p$). Let us also introduce the diagonal space h_p^d, defined as the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences. Recall that $\ell_p(L_p(\mathcal{M}))$ is the space of all sequences $a = (a_n)_{n\geq 0}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n=0}^{\infty} \|a_n\|_p^p\Big)^{1/p} < \infty,$$

with the usual modification for $p = \infty$. The conditioned Hardy space of noncommutative martingales is defined by

$$h_p = \begin{cases} h_p^d + h_p^c + h_p^r & \text{for} \quad 1 \leq p < 2 \\ h_p^d \cap h_p^c \cap h_p^r & \text{for} \quad 2 \leq p < \infty \end{cases}.$$

It was proved in [20] that for each n and $0 < p \leq \infty$, there exists an isometric right $\mathcal{M}_n$-module map $u_{n,p} : L_p(\mathcal{M}; \mathcal{E}_n) \to L_p(\mathcal{M}_n; \ell_2^c)$ with complemented range such that

$$u_{n,p}(x)^* u_{n,q}(y) = \mathcal{E}_n(x^* y), \tag{3.3.1}$$

for all $x \in L_p(\mathcal{M}; \mathcal{E}_n)$ and $y \in L_q(\mathcal{M}; \mathcal{E}_n)$. More precisely, for $0 < p < \infty$ there exists a contractive projection $\mathcal{Q}_{n,p}$ defined from $L_p(\mathcal{M}_n; \ell_2^c)$ onto the image of $u_{n,p}$ such that for all $\xi \in L_p(\mathcal{M}_n; \ell_2^c)$

$$\mathcal{Q}_{n,p}(\xi)^* \mathcal{Q}_{n,p}(\xi) \leq \xi^* \xi. \tag{3.3.2}$$

For $1 < p < \infty$ we know that

$$\mathcal{Q}_{n,p}^* = \mathcal{Q}_{n,p'}. \tag{3.3.3}$$

In the sequel for the sake of simplicity we will drop the subscript p in $u_{n,p}$ and $Q_{n,p}$. This proves that h_p^c isometrically embeds into $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ via the map

$$u : \begin{cases} h_p^c & \longrightarrow & L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2)) \\ x & \longmapsto & \displaystyle\sum_{n\geq 0} e_{n,0} \otimes u_{n-1}(d_n(x)) \end{cases}.$$

Furthermore, h_p^c is a complemented subspace of $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ for $1 < p < \infty$. Indeed, we can define a projection

$$P : L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2)) \to h_p^c$$

as follows. For $\xi = \sum_n e_{n,0} \otimes \xi_n \in L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$, for all $n \geq 0$ we have $\mathcal{E}_{n-1}(\xi_n) \in L_p(\mathcal{M}_{n-1}; \ell_2^c(\mathbb{N}))$. We may apply the projection $\mathcal{Q}_{n-1}$ and obtain for each n an element $y_n \in L_p(\mathcal{M})$ satisfying

$$\mathcal{Q}_{n-1}(\mathcal{E}_{n-1}(\xi_n)) = u_{n-1}(y_n). \tag{3.3.4}$$

Then we set

$$P(\xi) = \sum_{n\geq 0} d_n(y_n).$$

It is clear that $P \circ u = id_{h_p^c}$, i.e., that P is a projection from $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ onto h_p^c. Moreover, we can show that this projection is bounded for $1 < p < \infty$.

Lemma 3.3.1. *Let $1 < p < \infty$. Then h_p^c is γ_p-complemented in $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$.*

Proof. Let $\xi = \sum_n e_{n,0} \otimes \xi_n \in L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$. First observe that for all $n \geq 0$ we have

$$\mathcal{E}_{n-1}|d_n(y_n)|^2 \leq \mathcal{E}_{n-1}|y_n|^2. \tag{3.3.5}$$

Indeed, for $n = 0$, since by convention $\mathcal{E}_{-1} = \mathcal{E}_0$ and $d_0(y_0) = \mathcal{E}_0(y_0)$, we have

$$\mathcal{E}_0|d_0(y_0)|^2 = |\mathcal{E}_0(y_0)|^2 \leq \mathcal{E}_0|y_0|^2.$$

For $n \geq 1$, we can write

$$\begin{aligned}\mathcal{E}_{n-1}|d_n(y_n)|^2 &= \mathcal{E}_{n-1}(|\mathcal{E}_n(y_n)|^2 - |\mathcal{E}_{n-1}(y_n)|^2)\\ &\leq \mathcal{E}_{n-1}(|\mathcal{E}_n(y_n)|^2) \leq \mathcal{E}_{n-1}(\mathcal{E}_n|y_n|^2) = \mathcal{E}_{n-1}|y_n|^2.\end{aligned}$$

Moreover by (3.3.4) and (3.3.2), we have for all $n \geq 0$

$$\mathcal{E}_{n-1}|y_n|^2 = |u_{n-1}(y_n)|^2 = |\mathcal{Q}_{n-1}(\mathcal{E}_{n-1}(\xi_n))|^2 \leq |\mathcal{E}_{n-1}(\xi_n)|^2. \tag{3.3.6}$$

Combining (3.3.5) with (3.3.6) we obtain

$$\mathcal{E}_{n-1}|d_n(P(\xi))|^2 = \mathcal{E}_{n-1}|d_n(y_n)|^2 \leq |\mathcal{E}_{n-1}(\xi_n)|^2, \quad \forall\, n \geq 0. \tag{3.3.7}$$

The noncommutative Stein inequality implies

$$\begin{aligned}\|P(\xi)\|_{h_p^c} &= \Big\|\Big(\sum_{n\geq 0} \mathcal{E}_{n-1}|d_n(y_n)|^2\Big)^{1/2}\Big)\|_p \leq \Big\|\Big(\sum_{n\geq 0} |\mathcal{E}_{n-1}(\xi_n)|^2\Big)^{1/2}\Big\|_p\\ &\leq \gamma_p\Big\|\Big(\sum_{n\geq 0} |\xi_n|^2\Big)^{1/2}\Big\|_p = \gamma_p\|\xi\|_{L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))}.\end{aligned}$$

□

Since $(L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2)))^* = L_{p'}(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ isometrically for $1 \leq p < \infty$, we deduce from Lemma 3.3.1 the following duality result.

Corollary 3.3.2. *Let* $1 < p < \infty$. *Then*

$$(h_p^c)^* = h_{p'}^c \quad \textit{with equivalent norms.}$$

Moreover,

$$\gamma_p^{-1}\|x\|_{h_{p'}^c} \leq \|x\|_{(h_p^c)^*} \leq \|x\|_{h_{p'}^c}.$$

Remark 3.3.3. Observe that for $1 < p \leq \infty$ we have $P = u^*$. Indeed, for $x \in h_p^c$ and $\xi \in L_{p'}(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ we may write

$$\begin{aligned}(P(\xi)|x) &= \sum_n \tau(d_n(y_n)^* d_n(x)) = \sum_n \tau(y_n^* d_n(x))\\ &= \sum_n \tau(\mathcal{E}_{n-1}(y_n^* d_n(x)))\\ &= \sum_n \tau(u_{n-1}(y_n)^* u_{n-1}(d_n(x))) \qquad \text{by (3.3.1)}\\ &= \sum_n \tau(\mathcal{Q}_{n-1}(\mathcal{E}_{n-1}(\xi_n))^* u_{n-1}(d_n(x))) \qquad \text{by (3.3.4)}\\ &= \sum_n \tau(\mathcal{E}_{n-1}(\xi_n)^* \mathcal{Q}_{n-1}(u_{n-1}(d_n(x)))) \qquad \text{by (3.3.3)}\\ &= \sum_n \tau(\mathcal{E}_{n-1}(\xi_n)^* u_{n-1}(d_n(x))) = \sum_n \tau(\xi_n^* u_{n-1}(d_n(x)))\\ &= (\xi|u(x)).\end{aligned}$$

The analogue of the Fefferman-Stein duality for the conditioned case was established independently in [21] and Chapter 1. For $2 < p \leq \infty$ we introduce

$$L_p^c mo = \{x \in L_2(\mathcal{M}) \ : \ \|x\|_{L_p^c mo} < \infty\},$$

where

$$\|x\|_{L_p^c mo} = \max\big(\|\mathcal{E}_0(x)\|_p \ , \ \|{\sup_n}^+ \mathcal{E}_n |x - x_n|^2\|_{p/2}^{1/2}\big).$$

For $p = \infty$ we denote this space by bmo^c.

Theorem 3.3.4. *Let $1 \leq p < 2$. Then*

$$(h_p^c)^* = L_{p'}^c mo \quad \textit{with equivalent norms.}$$

Moreover,

$$\nu_p \|x\|_{L_{p'}^c mo} \leq \|x\|_{(h_p^c)^*} \leq \sqrt{2} \|x\|_{L_{p'}^c mo},$$

where ν_p remains bounded as $p \to 1$.

Combining these two latter results we obtain

Proposition 3.3.5. *Let $2 < p < \infty$. Then*

$$h_p^c = L_p^c mo \quad \textit{with equivalent norms.}$$

Observe that we can extend Lemma 3.3.1 to the case $p = \infty$ in the following sense.

Lemma 3.3.6. *Let $2 < p \leq \infty$. Then $P : L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2)) \to L_p^c mo$ is bounded.*

Proof. Let $\xi = \sum_n e_{n,0} \otimes \xi_n \in L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$ and $x = P(\xi)$. On the one hand, by (3.3.7) for $n = 0$ we have

$$\|\mathcal{E}_0(x)\|_p \leq \|\mathcal{E}_0(\xi_0)\|_p \leq \|\xi_0\|_p = \|(|\xi_0|^2)^{1/2}\|_p \leq \Big\| \Big(\sum_{n \geq 0} |\xi_n|^2 \Big)^{1/2} \Big\|_p = \|\xi\|_{L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))}.$$

On the other hand, note that by (3.3.7), for each $n \geq 0$ we have

$$\begin{aligned} \mathcal{E}_n |x - x_n|^2 &= \mathcal{E}_n \Big(\sum_{k > n} \mathcal{E}_{k-1} |d_k(x)|^2 \Big) \leq \mathcal{E}_n \Big(\sum_{k > n} |\mathcal{E}_{k-1}(\xi_k)|^2 \Big) \\ &\leq \mathcal{E}_n \Big(\sum_{k > n} \mathcal{E}_{k-1} |\xi_k|^2 \Big) = \mathcal{E}_n \Big(\sum_{k > n} |\xi_k|^2 \Big) \\ &\leq \mathcal{E}_n \Big(\sum_{k \geq 0} |\xi_k|^2 \Big). \end{aligned} \tag{3.3.8}$$

Since $1 < \frac{p}{2} \leq \infty$, the noncommutative Doob inequality gives

$$\|{\sup_n}^+ \mathcal{E}_n |x - x_n|^2\|_{p/2} \leq \Big\| {\sup_n}^+ \mathcal{E}_n \Big(\sum_{k \geq 0} |\xi_k|^2 \Big) \Big\|_{p/2} \leq \delta_{p/2} \Big\| \sum_{k \geq 0} |\xi_k|^2 \Big\|_{p/2} = \delta_{p/2} \|\xi\|^2_{L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))}.$$

Thus we get

$$\|P(\xi)\|_{L_p^c mo} = \|x\|_{L_p^c mo} = \max\big(\|\mathcal{E}_0(x)\|_p \ , \ \|{\sup_n}^+ \mathcal{E}_n |x - x_n|^2\|_{p/2}^{1/2}\big) \leq \max(1, \delta_{p/2}^{1/2}) \|\xi\|_{L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))}.$$

$\square$

We end this subsection with the noncommutative Burkholder inequalities proved in [24].

Theorem 3.3.7. *Let $1 < p < \infty$. Then*

$$L_p(\mathcal{M}) = h_p \quad \textit{with equivalent norms.}$$

Moreover,

$$\kappa_p^{-1}\|x\|_{h_p} \leq \|x\|_p \leq \eta_p\|x\|_{h_p}.$$

Remark 3.3.8. It is important to note that η_p remains bounded as $p \to 1$, i.e., for $p = 1$ we have a bounded inclusion $h_1 \subset L_1(\mathcal{M})$.

3.3.2 Definitions of $\widehat{\mathsf{h}}_p^c$, h_p^c and basic properties

As in the previous section, we fix an ultrafilter $\mathcal{U}$. For $\sigma \in \mathcal{P}_{\text{fin}}([0,1])$ and $x \in \mathcal{M}$, we define the finite conditioned bracket

$$\langle x, x\rangle_\sigma = \sum_{t\in\sigma} \mathcal{E}_{t^-}|d_t^\sigma(x)|^2$$

(recalling our convention that $\mathcal{E}_{0^-} = \mathcal{E}_0$). Observe that $\|\langle x, x\rangle_\sigma\|_{p/2}^{1/2} = \|x\|_{h_p^c(\sigma)}$, where $h_p^c(\sigma)$ denotes the noncommutative conditioned Hardy space with respect to the discrete filtration $(\mathcal{M}_t)_{t\in\sigma}$. Hence the noncommutative Burkholder inequalities recalled in Theorem 3.3.7 and the Hölder inequality imply for each finite partition σ and $x \in \mathcal{M}$

$$\begin{array}{ccccccc} \eta_p^{-1}\|x\|_p & \leq & \|\langle x, x\rangle_\sigma\|_{p/2}^{1/2} & \leq & \|x\|_2 & \text{for } 1 \leq p < 2 \\ \|x\|_2 & \leq & \|\langle x, x\rangle_\sigma\|_{p/2}^{1/2} & \leq & \kappa_p\|x\|_p & \text{for } 2 \leq p < \infty \end{array}. \tag{3.3.9}$$

Then, adapting the discussion detailed in subsection 3.2.2, for $x \in \mathcal{M}$ and $1 \leq p < \infty$ we may define

$$\langle x, x\rangle_{\mathcal{U}} = \mathcal{E}_{\mathcal{U}}((\langle x, x\rangle_\sigma)^\bullet) \quad , \quad \|x\|_{\widehat{\mathsf{h}}_p^c} = \|\langle x, x\rangle_{\mathcal{U}}\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{\mathsf{h}_p^c} = \lim_{\sigma,\mathcal{U}} \|x\|_{h_p^c(\sigma)}.$$

The properties of the conditional expectation $\mathcal{E}_{\mathcal{U}}$ imply the analogue of (3.2.3)

$$\begin{array}{ccccccccc} \eta_p^{-1}\|x\|_p & \leq & \|x\|_{\mathsf{h}_p^c} & \leq & \|x\|_{\widehat{\mathsf{h}}_p^c} & \leq & \|x\|_2 & \text{for } 1 \leq p < 2 \\ \|x\|_2 & \leq & \|x\|_{\widehat{\mathsf{h}}_p^c} & \leq & \|x\|_{\mathsf{h}_p^c} & \leq & \kappa_p\|x\|_p & \text{for } 2 \leq p < \infty \end{array}. \tag{3.3.10}$$

Hence $\|\cdot\|_{\widehat{\mathsf{h}}_p^c}$ and $\|\cdot\|_{\mathsf{h}_p^c}$ define two (quasi)norms on $\mathcal{M}$. As for $\widehat{\mathcal{H}}_p^c$ and $\mathcal{H}_p^c$, these (quasi)norms a priori depend on the choice of the ultrafilter $\mathcal{U}$. We will show that they actually do not, up to equivalent norm, and simply denote $\|\cdot\|_{\widehat{\mathsf{h}}_p^c}$ and $\|\cdot\|_{\mathsf{h}_p^c}$.

Definition 3.3.9. *Let $1 \leq p < \infty$. We define the spaces $\widehat{\mathsf{h}}_p^c$ and h_p^c as the completion of $\mathcal{M}$ with respect to the (quasi)norms $\|\cdot\|_{\widehat{\mathsf{h}}_p^c}$ and $\|\cdot\|_{\mathsf{h}_p^c}$ respectively.*

As for $\widehat{\mathcal{H}}_p^c$, we may equip $\widehat{\mathsf{h}}_p^c$ with an L_p $\mathcal{M}$-module structure and show that $\|\cdot\|_{\widehat{\mathsf{h}}_p^c}$ is a norm for $1 \leq p < \infty$.

Remark 3.3.10. In this case we also have

$$\widehat{\mathsf{h}}_p^c = \begin{cases} \overline{L_2(\mathcal{M})}^{\|\cdot\|_{\widehat{\mathsf{h}}_p^c}} & \text{for } 1 \leq p < 2 \\ \overline{L_p(\mathcal{M})}^{\|\cdot\|_{\widehat{\mathsf{h}}_p^c}} & \text{for } 2 \leq p < \infty \end{cases} \quad \text{and} \quad \mathsf{h}_p^c = \begin{cases} \overline{L_2(\mathcal{M})}^{\|\cdot\|_{\mathsf{h}_p^c}} & \text{for } 1 \leq p < 2 \\ \overline{L_p(\mathcal{M})}^{\|\cdot\|_{\mathsf{h}_p^c}} & \text{for } 2 \leq p < \infty \end{cases}.$$

In the conditioned case we still have some monotonicity properties of the discrete norms, but the monotonicity is inversed. That is an important difference with the $\mathcal{H}_p^c$-case.

Lemma 3.3.11. *Let $1 \le p < \infty$ and $\sigma \in \mathcal{P}_{\text{fin}}([0,1])$.*

(i) Let $1 \le p < 2$. Let $\sigma^1, \cdots, \sigma^M$ be partitions contained in σ, let $(\alpha_m)_{1\le m\le M}$ be a sequence of positive numbers such that $\sum_m \alpha_m = 1$, and let $x^1, \cdots, x^M \in L_2(\mathcal{M})$. Then for $x = \sum_m \alpha_m x^m$ we have

$$\|x\|_{h_p^c(\sigma)} \le 2^{1/p} \Big\| \sum_{m=1}^M \alpha_m \langle x^m, x^m\rangle_{\sigma^m} \Big\|_{p/2}^{1/2}.$$

In particular for $x \in L_2(\mathcal{M})$ and $\sigma \subset \sigma'$ we have

$$\|x\|_{h_p^c(\sigma')} \le 2^{1/p} \|x\|_{h_p^c(\sigma)}.$$

Hence

$$\inf_\sigma \|x\|_{h_p^c(\sigma)} \le \|x\|_{\mathsf{h}_p^c} \le 2^{1/p} \inf_\sigma \|x\|_{h_p^c(\sigma)}.$$

(ii) Let $2 \le p < \infty$. Let $\sigma^1, \cdots, \sigma^M$ be partitions containing σ, let $(\alpha_m)_{1\le m\le M}$ be a sequence of positive numbers such that $\sum_m \alpha_m = 1$, and let $x^1, \cdots, x^M \in L_p(\mathcal{M})$. Then for $x = \sum_m \alpha_m x^m$ we have

$$\|x\|_{h_p^c(\sigma)} \le \delta'^{1/2}_{p/2} \Big\| \sum_{m=1}^M \alpha_m \langle x^m, x^m\rangle_{\sigma^m} \Big\|_{p/2}^{1/2}.$$

In particular for $x \in L_p(\mathcal{M})$ and $\sigma \subset \sigma'$ we have

$$\|x\|_{h_p^c(\sigma)} \le \delta'^{1/2}_{p/2} \|x\|_{h_p^c(\sigma')}.$$

Hence

$$\delta'^{-1/2}_{p/2} \sup_\sigma \|x\|_{h_p^c(\sigma)} \le \|x\|_{\mathsf{h}_p^c} \le \sup_\sigma \|x\|_{h_p^c(\sigma)}.$$

Proof. We first consider $1 \le p < 2$. On the one hand, the operator convexity of $|\cdot|^2$ yields

$$\|x\|_{h_p^c(\sigma)}^2 = \Big\| \sum_{s\in\sigma} \mathcal{E}_{s^-} \Big| \sum_m \alpha_m d_s^\sigma(x^m) \Big|^2 \Big\|_{p/2} \le \Big\| \sum_{m,s\in\sigma} \alpha_m \mathcal{E}_{s^-} |d_s^\sigma(x^m)|^2 \Big\|_{p/2}.$$

On the other hand, for $1 \le m \le M$ and $t \in \sigma^m$ fixed we denote by I_t the collection of $s \in \sigma$ such that $t^- \le s^- < s \le t$. Then for m fixed, $\bigcup_{t\in\sigma^m} I_t = \sigma$. Note that for $1 \le m \le M$ and $t \in \sigma^m$, we can split up the interval $[t^-, t]$ in the subintervals $[s^-, s]$ with $s \in I_t$ and by the martingale property (and $t^- \le s^-$) we have

$$\mathcal{E}_{t^-} |d_t^{\sigma^m}(x^m)|^2 = \mathcal{E}_{t^-} \Big| \sum_{s\in I_t} d_s^\sigma(x^m) \Big|^2 = \mathcal{E}_{t^-} \Big(\sum_{s\in I_t} \mathcal{E}_{s^-} |d_s^\sigma(x^m)|^2 \Big). \tag{3.3.11}$$

Then (3.3.11) implies

$$\begin{aligned} \sum_m \alpha_m \langle x^m, x^m\rangle_{\sigma^m} &= \sum_m \alpha_m \sum_{t\in\sigma^m} \mathcal{E}_{t^-} \Big(\sum_{s\in I_t} \mathcal{E}_{s^-} |d_s^\sigma(x^m)|^2 \Big) \\ &= \sum_{m,s\in\sigma} \mathcal{E}_{t_m(s)^-} \big(\alpha_m \mathcal{E}_{s^-} |d_s^\sigma(x^m)|^2 \big), \end{aligned}$$

where $t_m(s)$ denotes the unique $t \in \sigma^m$ which satisfies $t^- \leq s^- < s \leq t$. We can rearrange the set $\{1, \cdots, M\} \times \sigma$ so that $\Big(\mathcal{M}_{t_m(s)^-}\Big)_{(m,s)}$ becomes an increasing sequence of von Neumann algebras. Thus we can apply the dual form of the reverse noncommutative Doob inequality for $0 < \frac{p}{2} < 1$ (Theorem 7.1 of [24]), and obtain

$$\begin{aligned}
\|x\|^2_{h^c_p(\sigma)} &\leq \Big\| \sum_{m,s\in\sigma} \alpha_m \mathcal{E}_{s^-} |d^\sigma_s(x^m)|^2 \Big\|_{p/2} \\
&\leq 2^{2/p} \Big\| \sum_{m,s\in\sigma} \mathcal{E}_{t_m(s)^-} \big(\alpha_m \mathcal{E}_{s^-} |d^\sigma_s(x^m)|^2\big) \Big\|_{p/2} \\
&= 2^{2/p} \Big\| \sum_m \alpha_m \langle x^m, x^m\rangle_{\sigma^m} \Big\|_{p/2}.
\end{aligned}$$

We now turn to assertion (ii). In this case, since $\sigma \subset \sigma^m$, for $t \in \sigma$ and m fixed we denote by I^m_t the collection of $s \in \sigma^m$ such that $t^- \leq s^- < s \leq t$. Then for m fixed, $\bigcup_{t\in\sigma} I_t = \sigma^m$. We observe that

$$\mathcal{E}_{t^-} |d^\sigma_t(x)|^2 = \sum_{m,l=1}^{M} \alpha_m \alpha_l \mathcal{E}_{t^-} (d^\sigma_t(x^m)^* d^\sigma_t(x^l)).$$

By Cauchy-Schwarz, we deduce that

$$\begin{aligned}
\|x\|_{h^c_p(\sigma)} &= \Big\| \sum_{t\in\sigma} \mathcal{E}_{t^-}(|d^\sigma_t(x)|^2) \Big\|_{p/2} \\
&\leq \Big\| \sum_{t\in\sigma,m,l} \alpha_l \alpha_m \mathcal{E}_{t^-}(|d^\sigma_t(x^m)|^2) \Big\|^{1/2}_{p/2} \Big\| \sum_{t\in\sigma,m,l} \alpha_l \alpha_m \mathcal{E}_{t^-}(|d^\sigma_t(x^l)|^2) \Big\|^{1/2}_{p/2} \\
&= \Big\| \sum_{t\in\sigma,m} \alpha_m \mathcal{E}_{t^-}(|d^\sigma_t(x^m)|^2) \Big\|_{p/2}.
\end{aligned}$$

Note that in the first term the summation over l disappears by using $\sum_l \alpha_l = 1$, and in the second one the summation over m disappears similarly. For $t \in \sigma$ and m as (3.3.11) we can write

$$\mathcal{E}_{t^-}(|d^\sigma_t(x^m)|^2) = \sum_{s\in I^m_t} \mathcal{E}_{t^-}(|d^{\sigma^m}_s(x^m)|^2).$$

By the dual version of the noncommutative Doob inequality for $1 \leq \frac{p}{2} < \infty$, we deduce that

$$\begin{aligned}
\Big\| \sum_{t\in\sigma,m} \alpha_m \mathcal{E}_{t^-}(|d^\sigma_t(x^m)|^2) \Big\|_{p/2} &= \Big\| \sum_{t\in\sigma,m,s\in I^m_t} \alpha_m \mathcal{E}_{t^-}(|d^{\sigma^m}_s(x^m)|^2) \Big\|_{p/2} \\
&= \Big\| \sum_{t\in\sigma} \mathcal{E}_{t^-}\Big(\sum_{m,s\in I^m_t} \alpha_m \mathcal{E}_{s^-}(|d^{\sigma^m}_s(x^m)|^2) \Big) \Big\|_{p/2} \\
&\leq \delta'_{p/2} \Big\| \sum_{t\in\sigma} \sum_{m,s\in I^m_t} \alpha_m \mathcal{E}_{s^-}(|d^{\sigma^m}_s(x^m)|^2) \Big\|_{p/2} \\
&= \Big\| \sum_{m=1}^{M} \alpha_m \langle x^m, x^m\rangle_{\sigma^m} \Big\|_{p/2}.
\end{aligned}$$

This ends the proof. $\square$

The independence (up to a constant) of h^c_p on $\mathcal{U}$ follows immediately.

Theorem 3.3.12. *For $1 \leq p < \infty$ the space h_p^c is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.*

Like for $\mathcal{H}_p^c$ in Lemma 3.2.14, we have the

Lemma 3.3.13. *Let $1 < p < \infty$. Then h_p^c is reflexive.*

3.3.3 Ultraproduct spaces and L_p-modules

As in subsection 3.2.3, we introduce the conditioned ultraproduct spaces and their regularized versions, into which we will isometrically embed the conditioned Hardy spaces defined in the previous subsection. We first define the ultraproduct of the column L_p-spaces with double indices.

Definition 3.3.14. *Let $1 \leq p < \infty$. We define*

$$\widetilde{\mathsf{k}}_p^c(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p(\mathcal{M}; \ell_2^c(\sigma \times \mathbb{N})) \quad \textit{and} \quad \mathsf{k}_p^c(\mathcal{U}) = \widetilde{\mathsf{k}}_p^c(\mathcal{U}) \cdot e_{\mathcal{U}},$$

where the point denotes the right modular action of $\widetilde{\mathcal{M}}_{\mathcal{U}}$ on $\widetilde{\mathsf{k}}_p^c(\mathcal{U})$.
For $p = \infty$, the definitions of $\widetilde{\mathsf{k}}_\infty^c(\mathcal{U})$ and $\mathsf{k}_\infty^c(\mathcal{U})$ are similar to that of $\widetilde{K}_\infty^c(\mathcal{U})$ and $K_\infty^c(\mathcal{U})$.

Then $\widetilde{\mathsf{k}}_p^c(\mathcal{U})$ is an L_p $\widetilde{\mathcal{M}}_{\mathcal{U}}$-module, $\mathsf{k}_p^c(\mathcal{U})$ is an L_p $\mathcal{M}_{\mathcal{U}}$-module and all the results we proved for $\widetilde{K}_p^c(\mathcal{U}), K_p^c(\mathcal{U})$ in subsection 3.2.3 still hold for $\widetilde{\mathsf{k}}_p^c(\mathcal{U}), \mathsf{k}_p^c(\mathcal{U})$. Let us now define the subspaces of these ultraproduct spaces consisting of martingales.

Definition 3.3.15. *Let $1 \leq p < \infty$. We define*

$$\widetilde{\mathsf{h}}_p^c(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} h_p^c(\sigma) \quad \textit{and} \quad \mathsf{h}_p^c(\mathcal{U}) = \overline{\bigcup_{\widetilde{p} > p} I_{\widetilde{p},p}(\widetilde{\mathsf{h}}_{\widetilde{p}}^c(\mathcal{U}))}^{\|\cdot\|_{\widetilde{\mathsf{h}}_p^c(\mathcal{U})}},$$

where $I_{\widetilde{p},p} : \widetilde{\mathsf{h}}_{\widetilde{p}}^c(\mathcal{U}) \to \widetilde{\mathsf{h}}_p^c(\mathcal{U})$ denotes the contractive ultraproduct of the componentwise inclusion maps.

Remark 3.3.16.
1. Observe that for $1 \leq p < \infty$, the map $i_{\mathcal{U}}$ extends to an isometric embedding from h_p^c into $\mathsf{h}_p^c(\mathcal{U})$.
2. Adapting the discussion of Remark 3.2.21 (2), we see that the map $\mathcal{E}_{\mathcal{U}}$ is well-defined and bounded from $\mathsf{h}_p^c(\mathcal{U})$ to $L_p(\mathcal{M})$ (resp. $L_2(\mathcal{M})$) for $1 \leq p < 2$ (resp. $2 \leq p < \infty$), but not necessarily faithful.

Let us now detail the isometric embedding of $\widetilde{\mathsf{h}}_p^c(\mathcal{U})$ into $\widetilde{\mathsf{k}}_p^c(\mathcal{U})$. For $1 \leq p < \infty$, we consider $U = (u_\sigma)^\bullet$, the ultraproduct map of the isometric inclusions

$$u_\sigma : \left\{ \begin{array}{ccc} h_p^c(\sigma) & \longrightarrow & L_p(\mathcal{M}; \ell_2^c(\sigma \times \mathbb{N})) \\ x & \longmapsto & \displaystyle\sum_{t \in \sigma} e_{t,0} \otimes u_{t^-}(d_t^\sigma(x)) \end{array} \right.$$

and $P = (P_\sigma)^\bullet$, the ultraproduct map of the projections

$$P_\sigma : \left\{ \begin{array}{ccc} L_p(\mathcal{M}; \ell_2^c(\sigma \times \mathbb{N})) & \longrightarrow & h_p^c(\sigma) \\ \xi & \longmapsto & P_\sigma(\xi) \end{array} \right.$$

defined in subsection 3.3.1. Note that

$$x = P(U(x)) \quad \text{for } x \in \widetilde{\mathsf{h}}_p^c(\mathcal{U}).$$

Then $U : \widetilde{\mathsf{h}}_p^c(\mathcal{U}) \to \widetilde{\mathsf{k}}_p^c(\mathcal{U})$ is still isometric and P is bounded for $1 < p < \infty$. Moreover U and P preserve the regularized spaces, i.e.,

$$U : \mathsf{h}_p^c(\mathcal{U}) \to \mathsf{k}_p^c(\mathcal{U}) \quad \text{and} \quad P : \mathsf{k}_p^c(\mathcal{U}) \to \mathsf{h}_p^c(\mathcal{U}).$$

Hence we get the complementation and duality results analoguous to Lemma 3.2.22 and Corollary 3.2.23 for the conditioned spaces. Observe that we have $P = U^*$ for $1 < p < \infty$.

We still have the following crucial density result for $\mathsf{h}_p^c(\mathcal{U})$.

Lemma 3.3.17. *Let $1 \leq p \leq 2$. Then $L_2(\mathcal{M}_\mathcal{U})$ is dense in $\mathsf{h}_p^c(\mathcal{U})$.*

Proof. The proof is similar to that of Lemma 3.2.24. By the same regularization process it suffices to consider $1 < p < 2$. Let $x \in \mathsf{h}_p^c(\mathcal{U})$ and $\varepsilon > 0$. Then $\xi = U(x) \in \mathsf{k}_p^c(\mathcal{U})$. Then for $\varepsilon > 0$ there exists $\eta \in \mathsf{k}_\infty^c(\mathcal{U})$ such that $\|\xi - \eta\|_{\widetilde{\mathsf{k}}_p^c(\mathcal{U})} < \varepsilon$. Hence $\eta \in \mathsf{k}_\infty^c(\mathcal{U}) \subset \mathsf{k}_2^c(\mathcal{U})$ and $a = P(\eta) \in L_2(\mathcal{M}_\mathcal{U})$ satisfies

$$\|x - a\|_{\widetilde{\mathsf{h}}_p^c(\mathcal{U})} = \|P(\xi) - P(\eta)\|_{\widetilde{\mathsf{h}}_p^c(\mathcal{U})} \leq \gamma_p \|\xi - \eta\|_{\widetilde{\mathsf{k}}_p^c(\mathcal{U})} < \gamma_p \varepsilon.$$

□

To sum up, for $1 \leq p < \infty$, h_p^c embeds isometrically into the L_p $\mathcal{M}_\mathcal{U}$-module $\mathsf{k}_p^c(\mathcal{U})$ via the map

$$U \circ i_\mathcal{U} : \mathsf{h}_p^c \xrightarrow{i_\mathcal{U}} \mathsf{h}_p^c(\mathcal{U}) \xrightarrow{U} \mathsf{k}_p^c(\mathcal{U}).$$

Similarly, we can embed isometrically the space $\widehat{\mathsf{h}}_p^c$ into the L_p $\mathcal{M}$-module $\widehat{\mathsf{k}}_p^c$ defined as follows. For $1 \leq p < \infty$ and $\xi, \eta \in \mathsf{k}_p^c(\mathcal{U})$, we consider the $L_{p/2}(\mathcal{M})$-valued inner product

$$\langle \xi, \eta \rangle_{\widehat{\mathsf{k}}_p^c(\mathcal{U})} = \mathcal{E}_\mathcal{U}(\langle \xi, \eta \rangle_{\widetilde{\mathsf{k}}_p^c(\mathcal{U})}) \in L_{p/2}(\mathcal{M}),$$

and the associated norm

$$\|\xi\|_{\widehat{\mathsf{k}}_p^c(\mathcal{U})} = \|\langle \xi, \xi \rangle_{\widehat{\mathsf{k}}_p^c(\mathcal{U})}\|_{p/2}^{1/2}.$$

Definition 3.3.18. *Let $1 \leq p < \infty$. We define*

$$\widehat{\mathsf{k}}_p^c(\mathcal{U}) = \begin{cases} \overline{\mathsf{k}_2^c(\mathcal{U})}^{\|\cdot\|_{\widehat{\mathsf{k}}_p^c(\mathcal{U})}} & \text{for } 1 \leq p < 2 \\ \overline{\mathsf{k}_p^c(\mathcal{U})}^{\|\cdot\|_{\widehat{\mathsf{k}}_p^c(\mathcal{U})}} & \text{for } 2 \leq p < \infty \end{cases}.$$

The map $U \circ i_\mathcal{U}$ defined for $x \in \mathcal{M}$ by

$$U \circ i_\mathcal{U}(x) = \Big(\sum_{t \in \sigma} e_{t,0} \otimes u_{t^-}(d_t^\sigma(x)) \Big)^\bullet$$

extends to an isometric embedding of $\widehat{\mathsf{h}}_p^c$ into $\widehat{\mathsf{k}}_p^c(\mathcal{U})$. By super-reflexivity of the L_p $\mathcal{M}$-module $\widehat{\mathsf{k}}_p^c(\mathcal{U})$, we deduce

Lemma 3.3.19. *Let $1 < p < \infty$. Then $\widehat{\mathsf{h}}_p^c$ is reflexive.*

3.3.4 $\widehat{\mathsf{h}}_p^c = \mathsf{h}_p^c$

In this subsection we show that in the conditioned case the two spaces $\widehat{\mathsf{h}}_p^c$ and h_p^c also coincide. In particular we will deduce that, up to an equivalent constant, these spaces do not depend on the choice of the ultrafilter $\mathcal{U}$.

Theorem 3.3.20. *Let $1 \leq p < \infty$. Then*

$$\mathsf{h}_p^c = \widehat{\mathsf{h}}_p^c \quad \textit{with equivalent norms.}$$

Theorem 3.3.12 immediately yields

Corollary 3.3.21. *For $1 \leq p < \infty$ the space $\widehat{\mathsf{h}}_p^c$ is independent of the choice of the ultrafilter $\mathcal{U}$, up to equivalent norm.*

We follow the same approach as in the proof of Theorem 3.2.28. We first consider the case $2 \leq p < \infty$ and prove the following complementation result.

Lemma 3.3.22. *Let $2 \leq p < \infty$. Then the map $\mathcal{E}_\mathcal{U} \circ P : \widehat{\mathsf{k}}_p^c(\mathcal{U}) \to \mathsf{h}_p^c$ is bounded.*

Proof. First note that since $\mathsf{k}_p^c(\mathcal{U})$ is dense in $\widehat{\mathsf{k}}_p^c(\mathcal{U})$, it suffices to consider

$$\xi = \Big(\sum_{t\in\sigma} e_{t,0} \otimes \xi_\sigma(t)\Big)^\bullet \in \mathsf{k}_p^c(\mathcal{U}) \quad \text{such that} \quad \|\xi\|_{\widehat{\mathsf{k}}_p^c(\mathcal{U})} = \|\langle \xi, \xi\rangle_{\widehat{\mathsf{k}}_p^c(\mathcal{U})}\|_{p/2}^{1/2} \leq 1,$$

where $\xi_\sigma(t) \in L_p(\mathcal{M}; \ell_2^c)$. Then $\mathcal{E}_\mathcal{U} \circ P(\xi)$ is well-defined, and $x = \mathcal{E}_\mathcal{U} \circ P(\xi) = \mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$, where

$$(x_\sigma)^\bullet = \Big(\sum_{t\in\sigma} d_t^\sigma(y_\sigma(t))\Big)^\bullet \in \mathsf{h}_p^c(\mathcal{U}).$$

Recall that for σ and $t \in \sigma$ fixed, $y_\sigma(t)$ is defined by

$$u_{t^-}(y_\sigma(t)) = \mathcal{Q}_{t^-}(\mathcal{E}_{t^-}(\xi_\sigma(t))).$$

Fix a partition σ_0. On the one hand, Lemma 3.3.11 yields for each $\sigma \supset \sigma_0$

$$\|x_\sigma\|_{h_p^c(\sigma_0)} \leq \delta_{p'/2}^{1/2} \|x_\sigma\|_{h_p^c(\sigma)} \leq C(p),$$

where $C(p)$ depends on $\|\xi\|_{\mathsf{k}_p^c(\mathcal{U})}$. We see that $(x_\sigma)_\sigma$ is uniformly bounded in the reflexive space $h_p^c(\sigma_0)$. Thus the weak-limit in $h_p^c(\sigma_0)$ exists and coincides with $\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$. Then we may approximate $\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)$ by convex combinations of the x_σ's in $h_p^c(\sigma_0)$-norm.
On the other hand, since $\langle \xi, \xi\rangle_{\widetilde{\mathsf{k}}_p^c(\mathcal{U})} \in L_{p/2}(\mathcal{M}_\mathcal{U})$, $\langle \xi, \xi\rangle_{\widehat{\mathsf{k}}_p^c(\mathcal{U})} = \mathcal{E}_\mathcal{U}(\langle \xi, \xi\rangle_{\widetilde{\mathsf{k}}_p^c(\mathcal{U})})$ coincides with the weak-limit of the elements $\langle \xi_\sigma, \xi_\sigma\rangle_{L_p(\mathcal{M};\ell_2^c(\sigma\times\mathbb{N}))} = \sum_{t\in\sigma} |\xi_\sigma(t)|^2$ in $L_{p/2}(\mathcal{M})$. Then, by considering the weak-limit of the elements $(x_\sigma, \sum_{t\in\sigma} |\xi_\sigma(t)|^2)$ in the space $h_p^c(\sigma_0) \oplus L_{p/2}(\mathcal{M})$, for $\varepsilon > 0$ we can find positive numbers $(\alpha_m)_{m=1}^M$ such that $\sum_m \alpha_m = 1$ and partitions $\sigma^1, \cdots, \sigma^M$ satisfying

$$\Big\|x - \sum_{m=1}^M \alpha_m x_{\sigma^m}\Big\|_{h_p^c(\sigma_0)} < \varepsilon \quad \text{and} \quad \Big\|\langle \xi, \xi\rangle_{\widehat{h}_p^c(\mathcal{U})} - \sum_{m=1}^M \alpha_m \sum_{t\in\sigma^m} |\xi_{\sigma^m}(t)|^2\Big\|_{p/2} < \varepsilon. \quad (3.3.12)$$

We first note that by the operator convexity of $|\cdot|^2$ and (3.3.11) we have

$$\begin{aligned}\Big\|\sum_{m=1}^{M}\alpha_m x_{\sigma^m}\Big\|_{h_p^c(\sigma_0)} &= \Big\|\sum_{t\in\sigma_0}\mathcal{E}_{t^-}\Big|\sum_{m=1}^{M}\alpha_m d_t^{\sigma_0}(x_{\sigma^m})\Big|^2\Big\|_{p/2}^{1/2}\\ &\leq \Big\|\sum_{t\in\sigma_0}\sum_{m=1}^{M}\alpha_m\mathcal{E}_{t^-}|d_t^{\sigma_0}(x_{\sigma^m})|^2\Big\|_{p/2}^{1/2}\\ &= \Big\|\sum_{t\in\sigma_0}\sum_{m=1}^{M}\alpha_m\mathcal{E}_{t^-}\Big(\sum_{s\in I_t^m}\mathcal{E}_{s^-}|d_s^{\sigma^m}(x_{\sigma^m})|^2\Big)\Big\|_{p/2}^{1/2}\\ &= \Big\|\sum_{t\in\sigma_0}\sum_{m=1}^{M}\alpha_m\mathcal{E}_{t^-}\Big(\sum_{s\in I_t^m}\mathcal{E}_{s^-}|d_s^{\sigma^m}(y_{\sigma^m}(s))|^2\Big)\Big\|_{p/2}^{1/2},\end{aligned}$$

where for m and $t\in\sigma_0$, I_t^m denotes the collection of $s\in\sigma^m$ such that $t^-\leq s^-<s\leq t$. Applying the dual Doob inequality we get

$$\begin{aligned}\Big\|\sum_{m=1}^{M}\alpha_m x_{\sigma^m}\Big\|_{h_p^c(\sigma_0)} &\leq {\delta'}_{p/2}^{1/2}\Big\|\sum_{t\in\sigma_0}\sum_{m=1}^{M}\alpha_m\sum_{s\in I_t^m}\mathcal{E}_{s^-}|d_s^{\sigma^m}(y_{\sigma^m}(s))|^2\Big\|_{p/2}^{1/2}\\ &= {\delta'}_{p/2}^{1/2}\Big\|\sum_{m=1}^{M}\alpha_m\sum_{s\in\sigma^m}\mathcal{E}_{s^-}|d_s^{\sigma^m}(y_{\sigma^m}(s))|^2\Big\|_{p/2}^{1/2}.\end{aligned}$$

Moreover, for m and $s\in\sigma^m$, by (3.3.7) we have

$$\mathcal{E}_{s^-}|d_s^{\sigma^m}(y_{\sigma^m}(s))|^2\leq|\mathcal{E}_{s^-}(\xi_{\sigma^m}(s))|^2.$$

Then the noncommutative Stein inequality implies

$$\Big\|\sum_{m=1}^{M}\alpha_m x_{\sigma^m}\Big\|_{h_p^c(\sigma_0)}\leq{\delta'}_{p/2}^{1/2}\gamma_p\Big\|\sum_{m=1}^{M}\alpha_m\sum_{s\in\sigma^m}|\xi_{\sigma^m}(s)|^2\Big\|_{p/2}^{1/2}.$$

Hence by (3.3.12) we obtain

$$\begin{aligned}\|x\|_{h_p^c(\sigma_0)} &\leq \varepsilon+\Big\|\sum_{m=1}^{M}\alpha_m x_{\sigma^m}\Big\|_{h_p^c(\sigma_0)}\leq\varepsilon+{\delta'}_{p/2}^{1/2}\gamma_p\Big\|\sum_{m=1}^{M}\alpha_m\sum_{s\in\sigma^m}|\xi_{\sigma^m}(s)|^2\Big\|_{p/2}^{1/2}\\ &\leq \varepsilon+{\delta'}_{p/2}^{1/2}\gamma_p(\varepsilon+\|\xi\|^2_{\mathsf{k}_p^c(\mathcal{U})})^{1/2}.\end{aligned}$$

Sending ε to 0 ends the proof. □

Proof of Theorem 3.3.20. For $2\leq p<\infty$, the proof is similar to that of Theorem 3.2.28 by replacing i and $\mathcal{D}$ by U and P respectively. Indeed, we use the fact in this case that for $x\in\mathcal{M}$ we have $x=\mathcal{E}_{\mathcal{U}}\circ P\circ U\circ i_{\mathcal{U}}(x)$ and $\|x\|_{\widehat{\mathsf{h}}_p^c}=\|U\circ i_{\mathcal{U}}(x)\|_{\widehat{\mathsf{k}}_p^c(\mathcal{U})}$. For $1\leq p<2$, we will use the same trick as in the proof of Lemma 3.2.32. Let us adapt this argument for $\widehat{\mathsf{h}}_p^c$. We fix $q>2$ and in the sequel we will consider $\widehat{\mathsf{h}}_p^c$ as the completion of $L_q(\mathcal{M})$. We consider the same index set

$$\mathcal{I}=\mathcal{P}_{\mathrm{fin}}(L_q(\mathcal{M}))\times\mathcal{P}_{\mathrm{fin}}([0,1])\times\mathbb{R}_+^*$$

and construct similarly the ultrafilter $\mathcal{V}$ on $\mathcal{I}$. As in subsection 3.2.4, for each $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$ we can find a sequence of positive numbers $(\alpha_m(i))_{m=1}^{M(i)}$ such that $\sum_m \alpha_m = 1$ and finite partitions $\sigma_i^1, \cdots, \sigma_i^{M(i)}$ containing σ_i and satisfying for all $x \in F$

$$\Big\| w\text{-}\lim_{\sigma,\mathcal{U}} \langle x, x\rangle_\sigma - \sum_{m=1}^{M(i)} \alpha_m(i) \langle x, x\rangle_{\sigma_i^m} \Big\|_{q/2} < \varepsilon.$$

In this case we consider the Hilbert space $\mathcal{H}_i = \ell_2\Big(\bigcup_{m,t\in\sigma_i^m} \{t\} \times \mathbb{N}\Big)$ equipped with the norm

$$\|(\xi_{m,t,j})_{1\le m\le M(i), t\in\sigma_i^m, j\in\mathbb{N}}\|_{\mathcal{H}_i} = \Big(\sum_{m=1}^{M(i)} \alpha_m(i) \sum_{t\in\sigma_i^m, j\in\mathbb{N}} |\xi_{m,t,j}|^2\Big)^{1/2}.$$

Then $\widehat{\mathsf{h}}_p^c$ embeds isometrically into $\prod_{\mathcal{V}} L_p(\mathcal{M}; \mathcal{H}_i^c)$ via the map $x \in L_q(\mathcal{M}) \mapsto \widetilde{x} = (\widetilde{x}(i))^\bullet$, where

$$\widetilde{x}(i) = \begin{cases} \displaystyle\sum_{m=1}^{M(i)} \sum_{t\in\sigma_i^m} e_{m,0} \otimes e_{t,0} \otimes u_{t^-}(d_t^{\sigma_i^m}(x)) & \text{if } i = (F, \sigma_i, \varepsilon) \text{ such that } x \in F \\ 0 & \text{otherwise} \end{cases}.$$

We will show that

$$(\widehat{\mathsf{h}}_p^c)^* \subset (\mathsf{h}_p^c)^*. \tag{3.3.13}$$

Let $\varphi \in (\widehat{\mathsf{h}}_p^c)^*$ be a functional of norm less than one. We may assume that φ is given by an element $\xi = (\xi(i))^\bullet \in \prod_{\mathcal{V}} L_{p'}(\mathcal{M}; \mathcal{H}_i^c)$ of norm less than one, with

$$\xi(i) = \sum_{m=1}^{M(i)} \sum_{t\in\sigma_i^m} e_{m,0} \otimes e_{t,0} \otimes \xi_{m,t}(i),$$

where $\xi_{m,t}(i) \in L_{p'}(\mathcal{M}; \ell_2^c(\mathbb{N}))$. Fix $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$ and $1 \le m \le M(i)$. We set

$$z_m(i) = P_{\sigma_i^m}(\xi_m(i)) \in L_{p'}(\mathcal{M}),$$

where $\xi_m(i) := \sum_{t\in\sigma_i^m} e_{m,0} \otimes e_{t,0} \otimes \xi_{m,t}(i) \in L_{p'}(\mathcal{M}; \ell_2^c(\sigma_i^m \times \mathbb{N}))$. Then we consider

$$z(i) = \sum_m \alpha_m(i) z_m(i) \in L_{p'}(\mathcal{M}).$$

We claim that $z(i)$ is a martingale in $L_{p'}^c mo(\sigma_i)$. The crucial point here is that by Lemma 3.3.6 the map $P_{\sigma_i^m} : L_{p'}(\mathcal{M}; \ell_2^c(\sigma_i^m \times \mathbb{N})) \to L_{p'}^c mo(\sigma_i^m)$ is bounded for $2 < p' \le \infty$. More precisely, on the one hand, (3.3.7) for $n = 0$ implies

$$|\mathcal{E}_0(z_m(i))|^2 \le |\mathcal{E}_0(\xi_{m,0}(i))|^2 \le \mathcal{E}_0 |\xi_{m,0}(i)|^2. \tag{3.3.14}$$

On the other hand, by (3.3.8) we have for all $s \in \sigma_i^m$ (and in particular for all $s \in \sigma_i \subset \sigma_i^m$)

$$\mathcal{E}_s |z_m(i) - \mathcal{E}_s(z_m(i))|^2 \le \mathcal{E}_s\Big(\sum_{t\in\sigma^m} |\xi_{m,t}(i)|^2\Big). \tag{3.3.15}$$

The operator convexity of the square function $|\cdot|^2$ yields

$$|\mathcal{E}_0(z(i))|^2 = \Big|\sum_m \alpha_m(i) \mathcal{E}_0(z_m(i))\Big|^2 \le \sum_m \alpha_m(i) |\mathcal{E}_0(z_m(i))|^2,$$

and for each $s \in \sigma_i$ we get

$$\mathcal{E}_s|z(i) - \mathcal{E}_s(z(i))|^2 = \mathcal{E}_s\Big|\sum_m \alpha_m(i)(z_m(i) - \mathcal{E}_s(z_m(i)))\Big|^2 \leq \sum_m \alpha_m(i)\mathcal{E}_s|z_m(i) - \mathcal{E}_s(z_m(i))|^2. \tag{3.3.16}$$

Then using (3.3.14) we obtain

$$|\mathcal{E}_0(z(i))|^2 \leq \mathcal{E}_0\Big(\sum_m \alpha_m(i)|\xi_{m,0}(i)|^2\Big),$$

and the contractivity of the conditional expectation $\mathcal{E}_0$ on $L_{p'/2}$ implies

$$\begin{aligned}\|\mathcal{E}_0(z(i))\|_{p'} &\leq \Big\|\mathcal{E}_0\Big(\sum_m \alpha_m(i)|\xi_{m,0}(i)|^2\Big)\Big\|_{p'/2}^{1/2} \leq \Big\|\sum_m \alpha_m(i)|\xi_{m,0}(i)|^2\Big\|_{p'/2}^{1/2}\\ &\leq \Big\|\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big\|_{p'/2}^{1/2} = \|\xi(i)\|_{L_{p'}(\mathcal{M};\mathcal{H}_i^c)}.\end{aligned}$$

Moreover (3.3.15) gives

$$\mathcal{E}_s|z(i) - \mathcal{E}_s(z(i))|^2 \leq \mathcal{E}_s\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big).$$

By the noncommutative Doob inequality we obtain

$$\begin{aligned}\|{\sup_{s\in\sigma_i}}^+\mathcal{E}_s|z(i) - \mathcal{E}_s(z(i))|^2\|_{p'/2} &\leq \Big\|{\sup_{s\in\sigma_i}}^+\mathcal{E}_s\Big(\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big)\Big\|_{p'/2}\\ &\leq \delta_{p'/2}\Big\|\sum_{m,t\in\sigma_i^m} \alpha_m(i)|\xi_{m,t}(i)|^2\Big\|_{p'/2}\\ &= \delta_{p'/2}\|\xi(i)\|^2_{L_{p'}(\mathcal{M};\mathcal{H}_i^c)}.\end{aligned}$$

Hence

$$\|z(i)\|_{L^c_{p'}mo(\sigma_i)} \leq \max(1, \delta_{p'/2}^{1/2})\|\xi(i)\|_{L_{p'}(\mathcal{M};\mathcal{H}_i^c)}.$$

In particular, we see that the family $(z(i))_i$ is uniformly bounded in $L_2(\mathcal{M})$. We set $z = w\text{-}\lim_{i,\mathcal{V}} z(i)$ in $L_2(\mathcal{M})$. We claim that $z \in (\mathsf{h}_p^c)^*$ with

$$\|z\|_{(\mathsf{h}_p^c)^*} \leq \sqrt{2}\max(1, \delta_{p'/2}^{1/2})\|\xi\|_{\prod_\mathcal{V} L_{p'}(\mathcal{M};\mathcal{H}_i^c)}. \tag{3.3.17}$$

By the density of $L_2(\mathcal{M})$ in h_p^c it suffices to estimate $|\tau(z^*x)|$ for all $x \in L_2(\mathcal{M})$ with $\|x\|_{\mathsf{h}_p^c} \leq 1$. Note that

$$\|x\|_{\mathsf{h}_p^c} = \lim_{i,\mathcal{V}} \|x\|_{h_p^c(\sigma_i)}. \tag{3.3.18}$$

Indeed, for all $\delta > 0$ and $x \in L_2(\mathcal{M})$, by definition of the h_p^c-norm we have

$$A_\delta = \{\sigma \in \mathcal{P}_{\text{fin}}([0,1]) \ : \ |\|x\|_{\mathsf{h}_p^c} - \|x\|_{h_p^c(\sigma)}| < \delta\} \in \mathcal{U}.$$

Hence the set $\mathcal{P}_{\text{fin}}(L_q(\mathcal{M})) \times A_\delta \times \mathbb{R}_+^* \in \mathcal{T} \times \mathcal{U} \times \mathcal{W} \subset \mathcal{V}$, and since

$$\mathcal{P}_{\text{fin}}(L_q(\mathcal{M})) \times A_\delta \times \mathbb{R}_+^* \subset \{i \in \mathcal{I} \ : \ |\|x\|_{\mathsf{h}_p^c} - \|x\|_{h_p^c(\sigma_i)}| < \delta\}$$

we deduce that the set in the right hand side is also in $\mathcal{V}$ for all δ, which proves (3.3.18). We conclude that for $x \in L_2(\mathcal{M})$ with $\|x\|_{\mathsf{h}_p^c} \leq 1$ we have

$$\begin{aligned}|\tau(z^*x)| &\leq \lim_{i,\mathcal{V}} |\tau(z(i)^*x)| \leq \sqrt{2} \lim_{i,\mathcal{V}} (\|z(i)\|_{L^c_{p'}mo(\sigma_i)} \|x\|_{h_p^c(\sigma_i)}) \\ &= \sqrt{2} (\lim_{i,\mathcal{V}} \|z(i)\|_{L^c_{p'}mo(\sigma_i)}) (\lim_{i,\mathcal{V}} \|x\|_{h_p^c(\sigma_i)}) \leq \sqrt{2} \max(1, \delta_{p'/2}^{1/2}) \|\xi\|_{\prod_{\mathcal{V}} L_{p'}(\mathcal{M};\mathcal{H}_i^c)} \|x\|_{\mathsf{h}_p^c} \\ &\leq \sqrt{2} \max(1, \delta_{p'/2}^{1/2}) \|\xi\|_{\prod_{\mathcal{V}} L_{p'}(\mathcal{M};\mathcal{H}_i^c)}.\end{aligned}$$

This proves (3.3.17). Finally, it remains to check that for all $x \in L_q(\mathcal{M})$, z satisfies

$$(\xi|\widetilde{x})_{\prod_{\mathcal{V}} L_{p'}(\mathcal{M};\mathcal{H}_i^c), \prod_{\mathcal{V}} L_p(\mathcal{M};\mathcal{H}_i^c)} = \tau(z^*x). \tag{3.3.19}$$

We first verify that for each $i = (F, \sigma_i, \varepsilon) \in \mathcal{I}$ such that $x \in F$ we have

$$(\xi(i)|\widetilde{x}(i))_{L_{p'}(\mathcal{M};\mathcal{H}_i^c), L_p(\mathcal{M};\mathcal{H}_i^c)} = \tau(z(i)^*x).$$

For all $1 \leq m \leq M(i)$, Remark 3.3.3 gives

$$\tau(z_m(i)^*x) = (P_{\sigma_i^m}(\xi_m(i))|x) = (\xi_m(i)|u_{\sigma_i^m}(x)) = \sum_{t \in \sigma_i^m} \tau\big(\xi_{m,t}(i)^* u_{t^-}(d_t^{\sigma_i^m}(x))\big).$$

Then

$$\tau(z(i)^*x) = \sum_{m=1}^{M(i)} \alpha_m(i) \tau(z_m(i)^*x) = \sum_{m=1}^{M(i)} \sum_{t \in \sigma_i^m} \alpha_m(i) \tau\big(\xi_{m,t}(i)^* u_{t^-}(d_t^{\sigma_i^m}(x))\big) = (\xi(i)|\widetilde{x}(i)).$$

As in the proof of (3.2.10), this is sufficient to show (3.3.19). The end of the proof of Theorem 3.3.20 is similar to that of Theorem 3.2.28. □

In the sequel, we will work with the space h_p^c.

3.3.5 Duality results

The aim of this subsection is to obtain the analoguous result of Corollary 3.3.2 in the continuous setting. In particular, thanks to the definition of h_p^c, this will imply that h_p^c embeds into $L_p(\mathcal{M})$ for $1 < p < 2$ and into $L_2(\mathcal{M})$ for $2 \leq p < \infty$. We will prove that this also holds true for $p = 1$. In fact, since the monotonicity for h_p^c is inverse to that of $\mathcal{H}_p^c$, the injectivity for $2 \leq p < \infty$ is a direct consequence of Lemma 3.3.11.

Proposition 3.3.23. *Let $2 \leq p < \infty$. Then*

(i) $\{x \in L_2(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^c} < \infty\}$ is complete with respect to the norm $\| \cdot \|_{\mathsf{h}_p^c}$.

(ii) h_p^c embeds into $L_2(\mathcal{M})$.

(iii) $\{x \in L_2(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^c} < \infty\} = (\mathsf{h}_{p'}^c)^*$ with equivalent norms.

Proof. Recall that in the conditioned case, by Lemma 3.3.11 the norms $\| \cdot \|_{h_p^c(\sigma)}$ are increasing in σ (up to a constant) for $2 \leq p < \infty$. Then the completeness of each discrete $h_p^c(\sigma)$-space yields the first assertion, and (ii) and (iii) follow as in the proof of Proposition 3.2.33. □

We introduce the following Banach space for technical reasons.

Definition 3.3.24. *Let $1 \leq p < \infty$. We define the space $\widetilde{\mathsf{h}}_p^c$ as the quotient space of $\mathsf{h}_p^c(\mathcal{U})$ by the kernel of the map $\mathcal{E}_\mathcal{U}$. The norm in $\widetilde{\mathsf{h}}_p^c$ is given by the usual quotient norm*

$$\|x\|_{\widetilde{\mathsf{h}}_p^c} = \inf_{x=\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)} \|(x_\sigma)^\bullet\|_{\mathsf{h}_p^c(\mathcal{U})} = \inf_{x=\mathcal{E}_\mathcal{U}((x_\sigma)^\bullet)} \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{h_p^c(\sigma)}.$$

Remark 3.3.25. By construction, $\widetilde{\mathsf{h}}_p^c$ embeds into $L_p(\mathcal{M})$ for $1 \leq p < 2$ and in $L_2(\mathcal{M})$ for $2 \leq p < \infty$.

Since the definitions of $\mathsf{h}_p^c, \widetilde{\mathsf{h}}_p^c$ are similar to that of $\mathcal{H}_p^c, \widetilde{\mathcal{H}}_p^c$, the duality between the spaces h_p^c and $\widetilde{\mathsf{h}}_{p'}^c$ still holds true in the conditioned case.

Proposition 3.3.26. *Let $1 < p < \infty$. Then*

$$(\mathsf{h}_p^c)^* = \widetilde{\mathsf{h}}_{p'}^c \quad \textit{with equivalent norms.}$$

Using this duality, we may show that in the conditioned case the spaces h_p^c and $\widetilde{\mathsf{h}}_p^c$ also coincide.

Proposition 3.3.27. *Let $1 \leq p < \infty$. Then*

$$\mathsf{h}_p^c = \widetilde{\mathsf{h}}_p^c \quad \textit{with equivalent norms.}$$

In particular, this proves that h_1^c embeds into $L_1(\mathcal{M})$. As for the proof of Proposition 3.2.37, we need the following boundedness of the conditional expectation $\mathcal{E}_\mathcal{U}$ for $1 \leq p \leq 2$.

Proposition 3.3.28. *Let $1 \leq p \leq 2$. Then $\mathcal{E}_\mathcal{U} : \mathsf{h}_p^c(\mathcal{U}) \to \mathsf{h}_p^c$ is a bounded projection.*

Since in this case the monotonicity is reversed, the proof is slightly different from that of Proposition 3.2.38, where we used the increasingness of $\|\cdot\|_{H_p^c(\sigma)}$ for $1 \leq p \leq 2$.

Proof. We first consider $x = (x_\sigma)^\bullet \in L_2(\mathcal{M}_\mathcal{U})$ such that $\|x\|_{\mathsf{h}_p^c(\mathcal{U})} = \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{h_p^c(\sigma)} < 1$. We may assume that $\|x_\sigma\|_{h_p^c(\sigma)} < 1$ for all σ. Hence $\mathcal{E}_\mathcal{U}(x)$ coincides with the weak-limit of the x_σ's in L_2. Thus for $\varepsilon > 0$ we may find positive numbers $(\alpha_m)_{m=1}^M$ such that $\sum_m \alpha_m = 1$ and partitions $\sigma^1, ..., \sigma^M$ such that

$$\|\mathcal{E}_\mathcal{U}(x) - \sum_m \alpha_m x_{\sigma^m}\|_2 < \varepsilon.$$

Then we deduce by Lemma 3.3.11 that

$$\begin{aligned}\|\mathcal{E}_\mathcal{U}(x)\|_{\mathsf{h}_p^c} &\leq \Big\|\mathcal{E}_\mathcal{U}(x) - \sum_m \alpha_m x_{\sigma^m}\Big\|_2 + \Big\|\sum_m \alpha_m x_{\sigma^m}\Big\|_{\mathsf{h}_p^c}\\ &\leq \varepsilon + \sum_m \alpha_m \|x_{\sigma^m}\|_{\mathsf{h}_p^c}\\ &\leq \varepsilon + 2^{1/p} \sum_m \alpha_m \|x_{\sigma^m}\|_{h_p^c(\sigma^m)} \leq \varepsilon + 2^{1/p}.\end{aligned}$$

Sending $\varepsilon \to 0$ we obtain that for all $x \in L_2(\mathcal{M}_\mathcal{U})$,

$$\|\mathcal{E}_\mathcal{U}(x)\|_{\mathsf{h}_p^c} \leq 2^{1/p} \|x\|_{\mathsf{h}_p^c(\mathcal{U})}.$$

We conclude the proof by using the density of $L_2(\mathcal{M}_\mathcal{U})$ in $\mathsf{h}_p^c(\mathcal{U})$ given by Lemma 3.3.17. $\square$

Proof of Proposition 3.3.27. The proof for $1 \leq p \leq 2$ is similar to that of Proposition 3.2.37, by using Lemma 3.3.17 and Lemma 3.3.28. The case $2 < p < \infty$ follows by the duality result established in Proposition 3.3.26 and the reflexivity of h_p^c. Indeed, for $2 < p < \infty$ we have

$$\mathsf{h}_p^c = (\widetilde{\mathsf{h}}_{p'}^c)^* = (\mathsf{h}_{p'}^c)^* = \widetilde{\mathsf{h}}_p^c$$

with equivalent norms. □

Combining Proposition 3.3.26 with Proposition 3.3.27 we obtain the expected duality result.

Theorem 3.3.29. *Let $1 < p < \infty$. Then*

$$(\mathsf{h}_p^c)^* = \mathsf{h}_{p'}^c \quad \textit{with equivalent norms.}$$

Proposition 3.2.33 then implies the

Corollary 3.3.30. *Let $2 \leq p < \infty$. Then $\mathsf{h}_p^c = \{x \in L_2(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^c} < \infty\}$.*

In particular, this shows that for $2 \leq p < \infty$, $L_p(\mathcal{M})$ is dense in the space $\{x \in L_2(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^c} < \infty\}$ with respect to the norm $\|\cdot\|_{\mathsf{h}_p^c}$. We also obtain the complementation of the h_p^c-spaces in the ultraproduct spaces $\mathsf{k}_p^c(\mathcal{U})$, and deduce that the conditioned Hardy spaces h_p^c form an interpolation scale for $1 < p < \infty$.

Corollary 3.3.31. *Let $1 < p < \infty$. Then h_p^c is complemented in $\mathsf{k}_p^c(\mathcal{U})$.*

Corollary 3.3.32. *Let $1 < p_1, p_2 < \infty$ and $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$\mathsf{h}_p^c = [\mathsf{h}_{p_1}^c, \mathsf{h}_{p_2}^c]_\theta \qquad \textit{with equivalent norms.}$$

We will see later, in Section 3.7, that this result still holds true for $p_1 = 1$.

3.3.6 Fefferman-Stein duality

This subsection deals with the analogue of the Fefferman-Stein duality for the conditioned Hardy spaces. First observe that in the discrete case, the space $L_p^c mo$ is simpler than the space $L_p^c MO$ for $2 < p \leq \infty$. Indeed, recall that for a finite partition σ and $x \in L_2(\mathcal{M})$ we have

$$\|x\|_{L_p^c MO(\sigma)} = \|{\sup_{t\in\sigma}}^+ \mathcal{E}_t|x - x_{t^-}|^2\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{L_p^c mo(\sigma)} = \max\left(\|\mathcal{E}_0(x)\|_p \ , \ \|{\sup_{t\in\sigma}}^+ \mathcal{E}_t|x - x_t|^2\|_{p/2}^{1/2}\right).$$

The crucial point is that the index "t^-", which depends on the partition σ, does not appear in the definition of $L_p^c mo(\sigma)$. Hence it is natural to introduce the following definition of $L_p^c\mathsf{mo}$ in the continuous setting.

Definition 3.3.33. *Let $2 < p \leq \infty$. We define*

$$L_p^c\mathsf{mo} = \{x \in L_2(\mathcal{M}) : \|x\|_{L_p^c\mathsf{mo}} < \infty\}$$

where

$$\|x\|_{L_p^c\mathsf{mo}} = \max\left(\|\mathcal{E}_0(x)\|_p \ , \ \|{\sup_{0\leq t\leq 1}}^+ \mathcal{E}_t|x - x_t|^2\|_{p/2}^{1/2}\right).$$

For $p = \infty$ we denote this space by bmo^c.

Recall that for a family $(x_t)_{0\leq t\leq 1}$ in $L_q(\mathcal{M})$, $1\leq q\leq\infty$, we define

$$\|\sup_{0\leq t\leq 1}{}^+ x_t\|_q = \|(x_t)_{0\leq t\leq 1}\|_{L_q(\mathcal{M};\ell_\infty([0,1]))} = \inf \|a\|_{2q}\sup_t \|y_t\|_\infty \|b\|_{2q},$$

where the infimum runs over all factorizations $x_t = ay_tb$ with $a,b\in L_{2q}(\mathcal{M})$ and $(y_t)\in \ell_\infty(L_\infty([0,1]))$. The space $L_p^c\mathsf{mo}$ obviously does not depend on $\mathcal{U}$. Note that by Proposition 2.1 of [26] we have

$$\sup_\sigma \|\sup_{t\in\sigma}{}^+\mathcal{E}_t|x-x_t|^2\|_{p/2}^{1/2} = \|\sup_{0\leq t\leq 1}{}^+\mathcal{E}_t|x-x_t|^2\|_{p/2}^{1/2},$$

thus we obtain

$$\|x\|_{L_p^c\mathsf{mo}} = \sup_\sigma \|x\|_{L_p^c mo(\sigma)}.$$

Since by definition $\|\cdot\|_{L_p^c mo(\sigma)}$ is increasing in σ, for $2<p\leq\infty$ we get

$$\|x\|_{L_p^c\mathsf{mo}} = \lim_{\sigma,\mathcal{U}} \|x\|_{L_p^c mo(\sigma)}$$

for every ultrafilter $\mathcal{U}$. This ensures that we really define a complete space.

We deduce from Proposition 3.3.5 that for $2<p<\infty$,

$$L_p^c\mathsf{mo} = \{x\in L_2(\mathcal{M})\ :\ \|x\|_{\mathsf{h}_p^c}<\infty\}\quad \text{with equivalent norms.}$$

Proposition 3.3.23 implies

Theorem 3.3.34. *Let $1\leq p<2$. Then*

$$(\mathsf{h}_p^c)^* = L_{p'}^c\mathsf{mo}\quad \textit{with equivalent norms.}$$

Moreover,

$$\nu_p^{-1}\|x\|_{L_{p'}^c\mathsf{mo}} \leq \|x\|_{(\mathsf{h}_p^c)^*} \leq \sqrt{2}\|x\|_{L_{p'}^c\mathsf{mo}}. \tag{3.3.20}$$

Proof. This follows easily from the discrete duality recalled in Theorem 3.3.4 and Lemma 3.3.11, by using an argument similar to that developed in the proof of Proposition 3.2.33 (iii). □

Moreover, Theorem 3.3.29 yields

Corollary 3.3.35. *Let $2<p<\infty$. Then*

$$L_p^c\mathsf{mo} = \mathsf{h}_p^c\quad \textit{with equivalent norms.}$$

As a consequence we have the following result, which characterizes the space $L_p^c\mathsf{mo}$ similarly to the definition of $L_p^c\mathcal{MO}$.

Lemma 3.3.36. *Let $2<p\leq\infty$. Then*

(i) The unit ball of $L_p^c\mathsf{mo}$ is equivalent to

$$\mathrm{B}_p = \{x = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma \ \textit{in}\ \ L_2\ :\ \lim_{\sigma,\mathcal{U}}\|x_\sigma\|_{L_p^c mo(\sigma)}\leq 1\}.$$

More precisely, we have

$$B_{L_p^c\mathsf{mo}} \subset \mathrm{B}_p \subset \sqrt{2}\nu_p B_{L_p^c\mathsf{mo}}.$$

(ii) Let $(x_\lambda)_\lambda$ be a sequence in $L_2(\mathcal{M})$ such that $\|x_\lambda\|_{L_p^c\mathsf{mo}} \leq 1$ for all λ and $x = w\text{-}\lim_\lambda x_\lambda$ in L_2. Then $x \in L_p^c\mathsf{mo}$ with $\|x\|_{L_p^c\mathsf{mo}} \leq \sqrt{2}\nu_p$.

Proof. It is clear that $B_{L_p^c\mathsf{mo}} \subset$ B. Conversely, let $x = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma$ in L_2 be such that $\lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{L_p^c mo(\sigma)} \leq 1$. By Theorem 3.3.34 and the density of $L_2(\mathcal{M})$ in $\mathsf{h}_{p'}^c$ we can write

$$\|x\|_{L_p^c\mathsf{mo}} \leq \nu_p \sup_{y \in L_2(\mathcal{M}), \|y\|_{\mathsf{h}_{p'}^c} \leq 1} |\tau(x^* y)|.$$

Note that for all $y \in L_2(\mathcal{M})$, $\|y\|_{\mathsf{h}_{p'}^c} \leq 1$ we have

$$\begin{aligned} |\tau(x^* y)| &\leq \lim_{\sigma,\mathcal{U}} |\tau(x_\sigma^* y)| \leq \sqrt{2} \lim_{\sigma,\mathcal{U}} \left(\|x_\sigma\|_{L_p^c mo(\sigma)} \|y\|_{h_{p'}^c(\sigma)} \right) \\ &= \sqrt{2} \big(\lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{L_p^c mo(\sigma)} \big) \big(\lim_{\sigma,\mathcal{U}} \|y\|_{h_{p'}^c(\sigma)} \big) \leq \sqrt{2}. \end{aligned}$$

Thus $x \in \sqrt{2}\nu_p B_{L_p^c\mathsf{mo}}$, and this proves (i). The proof of (ii) is similar to that of Corollary 3.2.47. ☐

3.4 The Davis decomposition and Burkholder inequalities for $1 < p < 2$

We continue our investigation of the Hardy spaces of noncommutative martingales in the continuous setting by studying some decompositions of $\mathcal{H}_p^c$ and $\mathcal{H}_p$ involving the conditioned Hardy space h_p^c.

3.4.1 The discrete case

We first recall the analogue of the Davis decomposition for noncommutative martingales in the discrete case, and we discuss a variant of this decomposition. Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration.

Observe that by combining the Burkholder-Gundy Theorem 3.2.7 with the Burkholder Theorem 3.3.7 we get

$$H_p = h_p \quad \text{with equivalent norms} \quad \text{for } 1 < p < \infty.$$

By a dual approach, it was proved in [21] and Chapter 1 that this equality still holds true for $p = 1$. Moreover, we can show a column version of this equality.

Theorem 3.4.1. *Let $1 \leq p < \infty$. Then*

$$H_p^c = \begin{cases} h_p^d + h_p^c & \text{for} \quad 1 \leq p < 2 \\ h_p^d \cap h_p^c & \text{for} \quad 2 \leq p < \infty \end{cases} \qquad \textit{with equivalent norms.}$$

Let us recall briefly the proof for $1 \leq p < 2$ (then we will deduce the case $2 < p < \infty$ by duality). The inclusion $h_p^d + h_p^c \subset H_p^c$ is easy, and the reverse inclusion is proved by duality. More precisely, we can show that

$$(h_p^d + h_p^c)^* = h_{p'}^d \cap L_{p'}^c mo \subset L_{p'}^c MO = (H_p^c)^*.$$

A close look at the dual spaces yields a stronger decomposition. Observe that for $2 < p' \leq \infty$ and $x \in L_2(\mathcal{M})$, by the triangle inequality in $L_{p'/2}(\mathcal{M};\ell_\infty)$ we can write

$$\Big\|{\sup_{n\geq 0}}^+ \mathcal{E}_n\Big(\sum_{k\geq n}|d_k(x)|^2\Big)\Big\|_{p'/2} \simeq \|{\sup_{n\geq 0}}^+ |d_n(x)|^2\|_{p'/2} + \Big\|{\sup_{n\geq 0}}^+ \mathcal{E}_n\Big(\sum_{k>n}|d_k(x)|^2\Big)\Big\|_{p'/2}.$$

Hence we get

$$\|x\|_{L^c_{p'}MO} \simeq \max\big(\|(d_n(x))_n\|_{L_{p'}(\mathcal{M};\ell^c_\infty)}\ ,\ \|x\|_{L^c_{p'}mo}\big). \tag{3.4.1}$$

Recall that for $2 < p' \leq \infty$, $L_{p'}(\mathcal{M};\ell^c_\infty)$ is defined in [20, 31] as the space of all sequences $x = (x_n)_{n\geq 0}$ in $L_{p'}(\mathcal{M})$ such that

$$\|(x_n)_{n\geq 0}\|_{L_{p'}(\mathcal{M};\ell^c_\infty)} = \|(|x_n|^2)_{n\geq 0}\|^{1/2}_{L_{p'/2}(\mathcal{M};\ell_\infty)} = \|{\sup_{n\geq 0}}^+ |x_n|^2\|^{1/2}_{p'/2} < \infty.$$

Note that a sequence $x = (x_n)_{n\geq 0}$ in $L_{p'}(\mathcal{M})$ belongs to $L_{p'}(\mathcal{M};\ell^c_\infty)$ if and only if there exist $a \in L_{p'}(\mathcal{M})$ and $y = (y_n)_{n\geq 0} \subset L_\infty(\mathcal{M})$ such that $x_n = y_n a$ for all $n \geq 0$. Moreover,

$$\|x\|_{L_{p'}(\mathcal{M};\ell^c_\infty)} = \inf\{\sup_{n\geq 0}\|y_n\|_\infty \|a\|_{p'}\},$$

where the infimum runs over all factorizations as above.

Inspired by the duality between $L_p(\mathcal{M};\ell_1)$ and $L_{p'}(\mathcal{M};\ell_\infty)$ proved in [20], we define its predual space $L_p(\mathcal{M};\ell^c_1)$ as follows. Let $1 \leq p < 2$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. A sequence $x = (x_n)_{n\geq 0}$ is in $L_p(\mathcal{M};\ell^c_1)$ if there exist $b_{k,n} \in L_2(\mathcal{M})$ and $a_{k,n} \in L_q(\mathcal{M})$ such that

$$x_n = \sum_{k\geq 0} b^*_{k,n} a_{k,n} \tag{3.4.2}$$

for all n and

$$\sum_{k,n\geq 0} |b_{k,n}|^2 \in L_1(\mathcal{M}), \qquad \sum_{k,n\geq 0} |a_{k,n}|^2 \in L_{q/2}(\mathcal{M}).$$

We equip $L_p(\mathcal{M};\ell^c_1)$ with the norm

$$\|x\|_{L_p(\mathcal{M};\ell^c_1)} = \inf\Big\{\Big(\sum_{k,n\geq 0}\|b_{k,n}\|_2^2\Big)^{1/2}\Big\|\Big(\sum_{k,n\geq 0}|a_{k,n}|^2\Big)^{1/2}\Big\|_q\Big\},$$

where the infimum is taken over all factorizations (3.4.2). In fact this space can be described in an easier way.

Lemma 3.4.2. *Let $1 \leq p < 2$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Then the unit ball of $L_p(\mathcal{M};\ell^c_1)$ is the set of all sequences $(b_n a_n)_{n\geq 0}$ such that*

$$\Big(\sum_{n\geq 0}\|b_n\|_2^2\Big)^{1/2}\Big\|\Big(\sum_{n\geq 0}|a_n|^2\Big)^{1/2}\Big\|_q \leq 1. \tag{3.4.3}$$

Proof. It is clear that a sequence $(b_n a_n)_{n\geq 0}$ satisfying (3.4.3) is in the unit ball of $L_p(\mathcal{M};\ell^c_1)$. Conversely, let $x = (x_n)_{n\geq 0}$ be such that $x_n = \sum_{k\geq 0} b^*_{k,n} a_{k,n}$ with

$$\Big(\sum_{k,n\geq 0}\|b_{k,n}\|_2^2\Big)^{1/2}\Big\|\Big(\sum_{k,n\geq 0}|a_{k,n}|^2\Big)^{1/2}\Big\|_q \leq 1.$$

We first set $a'_n = \Big(\sum_{k\geq 0}|a_{k,n}|^2\Big)^{1/2}$. By approximation, we may assume that the a'_n's are invertible. Then considering

$$v_{k,n} = a_{k,n}{a'}_n^{-1} \quad \text{and} \quad b'_n = \sum_{k\geq 0} b^*_{k,n} v_{k,n},$$

we can write $x_n = b'_n a'_n$ for all $n \geq 0$. Moreover,

$$\Big\|\Big(\sum_{n\geq 0}|a'_n|^2\Big)^{1/2}\Big\|_q = \Big\|\Big(\sum_{n,k\geq 0}|a_{k,n}|^2\Big)^{1/2}\Big\|_q$$

and since $\sum_{k\geq 0}|v_{k,n}|^2 = 1$ we get

$$\begin{aligned}
\sum_{n\geq 0}\|b'_n\|_2^2 &= \sum_{n\geq 0}\Big\|\sum_{k\geq 0} b^*_{k,n}v_{k,n}\Big\|_2^2 \\
&\leq \sum_{n\geq 0}\Big\|\Big(\sum_{k\geq 0} b^*_{k,n}b_{k,n}\Big)^{1/2}\Big\|_2^2\Big\|\Big(\sum_{k\geq 0} v^*_{k,n}v_{k,n}\Big)^{1/2}\Big\|_\infty^2 \\
&= \sum_{k,n\geq 0}\|b_{k,n}\|_2^2.
\end{aligned}$$

Hence (a'_n) and (b'_n) satisfy (3.4.3). □

Remark 3.4.3. This implies that we have a bounded map

$$\begin{cases} L_p(\mathcal{M};\ell_1^c) & \longrightarrow & L_p(\mathcal{M};\ell_2^c) \\ (b_na_n)_{n\geq 0} & \longmapsto & \sum_{n\geq 0} e_{n,0}\otimes b_na_n \end{cases}.$$

Indeed, we can write

$$\sum_n e_{n,0}\otimes b_na_n = \Big(\sum_n e_{n,n}\otimes b_n\Big)\Big(\sum_n e_{n,0}\otimes a_n\Big)$$

and the Hölder inequality gives for $\frac{1}{p} = \frac{1}{2}+\frac{1}{q}$

$$\Big\|\sum_n e_{n,0}\otimes b_na_n\Big\|_p \leq \Big\|\sum_n e_{n,n}\otimes b_n\Big\|_2\Big\|\sum_n e_{n,0}\otimes a_n\Big\|_q = \Big(\sum_n\|b_n\|_2^2\Big)^{1/2}\Big\|\Big(\sum_n|a_n|^2\Big)^{1/2}\Big\|_q.$$

We can now state the following duality. Its proof is similar to the duality between $L_p(\mathcal{M};\ell_1)$ and $L_{p'}(\mathcal{M};\ell_\infty)$ in Proposition 3.6 of [20]. The main ingredient is a standard application of the Grothendieck-Pietsch version of the Hahn-Banach Theorem.

Proposition 3.4.4. *Let $1 \leq p < 2$. Then*

$$(L_p(\mathcal{M};\ell_1^c))^* = L_{p'}(\mathcal{M};\ell_\infty^c) \quad \textit{isometrically.}$$

Let $h_p^{1_c}$ (resp. $h_{p'}^{\infty_c}$) be the subspace of $L_p(\mathcal{M};\ell_1^c)$ (resp. $L_{p'}(\mathcal{M};\ell_\infty^c)$) consisting of all martingale difference sequences.

Lemma 3.4.5. *Let $1 \leq p \leq \infty$.*

(i) For $1 \leq p < 2$, $h_p^{1_c}$ is a complemented subspace of $L_p(\mathcal{M};\ell_1^c)$.

(ii) For $2 < p \leq \infty$, $h_p^{\infty_c}$ *is a complemented subspace of* $L_p(\mathcal{M}; \ell_\infty^c)$.

Proof. We first show that the Stein projection

$$\mathcal{D}((x_n)_{n\geq 0}) = (d_n(x_n))_{n\geq 0}$$

is bounded on $L_p(\mathcal{M}; \ell_1^c)$ for $1 \leq p < 2$. Let $(x_n)_n$ be in the unit ball of $L_p(\mathcal{M}; \ell_1^c)$ and let $x_n = b_n a_n$ be the decomposition of x_n given by Lemma 3.4.2. Then for each n we can write

$$\mathcal{E}_n(x_n) = u_n(b_n^*)^* u_n(a_n) = \sum_{n,k} u_n(b_n^*)(k)^* u_n(a_n)(k),$$

where $u_n(b_n^*)(k) \in L_2(\mathcal{M})$ and $u_n(a_n)(k) \in L_q(\mathcal{M})$. On the one hand, the trace preserving property of the conditional expectation gives

$$\sum_{n,k} \|u_n(b_n^*)(k)\|_2^2 = \sum_n \tau(\mathcal{E}_n(b_n^* b_n)) = \sum_n \|b_n\|_2^2.$$

On the other hand, since we have $2 \leq q < \infty$ for $1 \leq p < 2$, the dual form of the Doob inequality yields

$$\Big\| \sum_{n,k} |u_n(a_n)(k)|^2 \Big\|_{q/2} = \Big\| \sum_n \mathcal{E}_n |a_n|^2 \Big\|_{q/2} \leq \delta'_{q/2} \Big\| \sum_n |a_n|^2 \Big\|_{q/2}.$$

Hence $(\mathcal{E}_n(x_n))_n \in L_p(\mathcal{M}; \ell_1^c)$ with $\|(\mathcal{E}_n(x_n))_n\|_{L_p(\mathcal{M};\ell_1^c)} \leq \delta'^{1/2}_{q/2}$, where $\delta'_{q/2} \approx q^2$ as $q \to \infty$, $p \to 2$. This shows that $h_p^{1_c}$ is $2\delta'^{1/2}_{q/2}$-complemented in $L_p(\mathcal{M}; \ell_1^c)$ for $1 \leq p < 2$.

For the second assertion, the noncommutative Doob inequality and the fact that $|\mathcal{E}_n(x_n)|^2 \leq \mathcal{E}_n |x_n|^2$ immediately imply that $h_p^{\infty_c}$ is $2\delta^{1/2}_{p/2}$-complemented in $L_p(\mathcal{M}; \ell_\infty^c)$. □

Combining Proposition 3.4.4 with Lemma 3.4.5 we get the duality between $h_p^{1_c}$ and $h_{p'}^{\infty_c}$.

Corollary 3.4.6. *Let* $1 \leq p < 2$. *Then*

$$(h_p^{1_c})^* = h_{p'}^{\infty_c} \quad \textit{with equivalent norms.}$$

Then (3.4.1) means that for $1 \leq p < 2$, we have by Corollary 3.4.6

$$(H_p^c)^* = L_{p'}^c MO = h_{p'}^{\infty_c} \cap L_{p'}^c mo = (h_p^{1_c} + h_p^c)^*.$$

This yields the following stronger Davis decomposition.

Theorem 3.4.7. *Let* $1 \leq p < 2$. *Then*

$$H_p^c = h_p^{1_c} + h_p^c \quad \textit{with equivalent norms.}$$

Remark 3.4.8.

1. Observe that by interpolation between the cases $p = 1$ and $p = 2$ we have a contractive inclusion $L_p(\mathcal{M}; \ell_1^c) \subset \ell_p(L_p(\mathcal{M}))$ for $1 \leq p \leq 2$. Thus, considering the martingale difference sequences, we get

$$h_p^{1_c} \subset h_p^d \quad \text{contractively for } 1 \leq p < 2.$$

Hence the decomposition of Theorem 3.4.7 is stronger than the usual decomposition stated in Theorem 3.4.1.

2. The advantage of working with the spaces $h_p^{1_c}$ is that, since $\mathcal{M}$ is finite, they satisfy the following regularity property

$$h_{\widetilde{p}}^{1_c} \subset h_p^{1_c} \quad \text{contractively for } 1 \leq p \leq \widetilde{p} < 2,$$

whereas the h_p^d spaces *do not*. However we loose the reflexivity property.

3.4.2 Definition of diagonal spaces for $1 \leq p < 2$ and basic properties

We fix an ultrafilter $\mathcal{U}$. For $x \in \mathcal{M}$ and $1 \leq p < 2$, whenever the limits exist, we define

$$\|x\|_{\mathsf{h}_p^d} = \lim_{\sigma,\mathcal{U}} \|x\|_{h_p^d(\sigma)} \quad \text{and} \quad \|x\|_{\mathsf{h}_p^{1c}} = \lim_{\sigma,\mathcal{U}} \|x\|_{h_p^{1c}(\sigma)}.$$

Observe that by interpolation between the cases $p = 1$ and $p = 2$ and Remark 3.4.8 we have

$$\frac{1}{2}\|x\|_p \leq \|x\|_{\mathsf{h}_p^d} \leq \|x\|_{\mathsf{h}_p^{1c}}.$$

Hence $\|\cdot\|_{\mathsf{h}_p^d}$ and $\|\cdot\|_{\mathsf{h}_p^{1c}}$ define two norms for $1 \leq p < 2$.
The discrete diagonal norms also satisfy some monotonicity properties.

Lemma 3.4.9. *Let $1 \leq p < 2$, $x \in \mathcal{M}$ and $\sigma \subset \sigma'$. Then*

(i) $\|x\|_{h_p^d(\sigma)} \leq 2\|x\|_{h_p^d(\sigma')}$. *Hence*

$$\|x\|_{\mathsf{h}_p^d} \leq \sup_\sigma \|x\|_{h_p^d(\sigma)} \leq 2\|x\|_{\mathsf{h}_p^d}.$$

(ii) $\|x\|_{h_p^{1c}(\sigma)} \leq \|x\|_{h_p^{1c}(\sigma')}$. *Hence*

$$\|x\|_{\mathsf{h}_p^{1c}} = \sup_\sigma \|x\|_{h_p^{1c}(\sigma)}.$$

Proof. Let $\sigma \subset \sigma'$. By interpolation between the cases $p = 1$ and $p = 2$ we have for $1 \leq p \leq 2$ and $t \in \sigma$

$$\|d_t^\sigma(x)\|_p = \Big\| \sum_{s\in I_t} d_s^{\sigma'}(x) \Big\|_p \leq 2\Big(\sum_{s\in I_t} \|d_s^{\sigma'}(x)\|_p^p \Big)^{\frac{1}{p}},$$

where I_t denotes the collection of $s \in \sigma'$ such that $t^- \leq s^- < s \leq t$. Thus

$$\|x\|_{h_p^d(\sigma)} \leq 2\|x\|_{h_p^d(\sigma')}.$$

For (ii), we show that for $\sigma \subset \sigma'$ we have a contractive map

$$\Sigma : \left\{ \begin{array}{ccc} L_p(\mathcal{M}; \ell_1^c(\sigma')) & \longrightarrow & L_p(\mathcal{M}; \ell_1^c(\sigma)) \\ (x_s)_{s\in\sigma'} & \longmapsto & (x_t)_{t\in\sigma} = \Big(\sum_{s\in I_t} x_s \Big)_{t\in\sigma} \end{array} \right. .$$

Since for $x \in \mathcal{M}$ we have $\Sigma((d_s^{\sigma'}(x))_{s\in\sigma'}) = (d_t^\sigma(x))_{t\in\sigma}$, this will yield the required result for h_p^{1c}. Let $x = (x_s)_{s\in\sigma'}$ be in the unit ball of $L_p(\mathcal{M}; \ell_1^c(\sigma'))$, then by Lemma 3.4.2 we may write $x_s = b_s a_s$ for all $s \in \sigma'$ with

$$\Big(\sum_{s\in\sigma'} \|b_s\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_{s\in\sigma'} |a_s|^2 \Big)^{1/2} \Big\|_q \leq 1,$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Then $\Sigma(x) = \Big(\sum_{s\in I_t} b_s a_s \Big)_{t\in\sigma}$ is of the form (3.4.2) with

$$\Big(\sum_{t\in\sigma, s\in I_t} \|b_s^*\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_{t\in\sigma, s\in I_t} |a_s|^2 \Big)^{1/2} \Big\|_q = \Big(\sum_{s\in\sigma'} \|b_s\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_{s\in\sigma'} |a_s|^2 \Big)^{1/2} \Big\|_q \leq 1.$$

Hence $\Sigma(x)$ is in the unit ball of $L_p(\mathcal{M}; \ell_1^c(\sigma))$. $\square$

Corollary 3.4.10. *Let $1 \leq p < 2$. Then the norms $\|\cdot\|_{\mathsf{h}_p^d}$ and $\|\cdot\|_{\mathsf{h}_p^{1_c}}$ do not depend on the choice of the ultrafilter $\mathcal{U}$, up to a constant.*

Definition 3.4.11. *Let $1 \leq p < 2$. We define*

$$\widetilde{\mathsf{h}}_p^d = \{x \in L_p(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^d} < \infty\} \quad \textit{and} \quad \widetilde{\mathsf{h}}_p^{1_c} = \{x \in L_p(\mathcal{M}) \ : \ \|x\|_{\mathsf{h}_p^{1_c}} < \infty\}.$$

Adapting the proof of Proposition 3.2.33 we can show that these define two Banach spaces. By Remark 3.4.8 (1) we have

$$\widetilde{\mathsf{h}}_p^{1_c} \subset \widetilde{\mathsf{h}}_p^d \quad \text{contractively for } 1 \leq p < 2.$$

For technical reasons these spaces are too large. Hence we need to introduce their regularized versions as follows. Note that by the regularity property of the $h_p^{1_c}(\sigma)$-spaces stated in Remark 3.4.8 and the fact that $\widetilde{\mathsf{h}}_p^{1_c}$ is a subspace of $L_p(\mathcal{M})$, we have

$$\widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c} \subset \widetilde{\mathsf{h}}_p^{1_c} \quad \text{contractively for } 1 \leq p \leq \widetilde{p} < 2.$$

Definition 3.4.12. *Let $1 \leq p < 2$. We define*

$$\mathsf{h}_p^d = \overline{L_2(\mathcal{M}) \cap \widetilde{\mathsf{h}}_p^d}^{\|\cdot\|_{\mathsf{h}_p^d}} \quad \textit{and} \quad \mathsf{h}_p^{1_c} = \overline{\bigcup_{\widetilde{p}>p} \widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}}^{\|\cdot\|_{\mathsf{h}_p^{1_c}}}.$$

Remark 3.4.13.

1. At this point it is not obvious that the set $L_2(\mathcal{M}) \cap \widetilde{\mathsf{h}}_p^d$ is non trivial. We will show later that this definition of h_p^d actually makes sense.

2. Note that for $1 \leq p \leq 2$ we have bounded inclusions

$$\mathsf{h}_p^d \subset \widetilde{\mathsf{h}}_p^d \subset L_p(\mathcal{M}) \quad \text{and} \quad \mathsf{h}_p^{1_c} \subset \widetilde{\mathsf{h}}_p^{1_c} \subset L_p(\mathcal{M}).$$

Since by Proposition 3.2.33 we have an injective map $\mathcal{H}_p^c \hookrightarrow L_p(\mathcal{M})$, this implies that the natural bounded maps

$$\mathsf{h}_p^d \hookrightarrow \mathcal{H}_p^c \quad \text{and} \quad \mathsf{h}_p^{1_c} \hookrightarrow \mathcal{H}_p^c$$

are injectives. Similarly, since Theorem 3.3.29 implies that the natural map $\mathsf{h}_p^c \hookrightarrow L_p(\mathcal{M})$ is injective, we deduce that the map

$$\mathsf{h}_p^c \hookrightarrow \mathcal{H}_p^c$$

is injective. Hence in what follows we will consider the spaces $\mathsf{h}_p^d, \mathsf{h}_p^{1_c}$ and h_p^c as subspaces of $\mathcal{H}_p^c$.

3.4.3 The Davis decomposition for ultraproduct spaces

We use the same approach as in subsection 3.2.7. We will first prove the Davis decomposition for the ultraproduct spaces, then for their regularized version. Let us introduce the ultraproduct spaces and their regularized versions associated to the diagonal spaces. Since the spaces $h_p^{1_c}(\sigma)$ are regular in the sense of Remark 3.4.8 (2), we may define the regularized spaces in the same way as we defined the spaces $\mathcal{H}_p^c(\mathcal{U})$ and $\mathsf{h}_p^c(\mathcal{U})$. However, the diagonal spaces $h_p^d(\sigma)$ are not regular. Hence we will introduce a different definition for the regularized space associated to h_p^d.

Definition 3.4.14. *Let $1 \leq p < 2$. We define*

(i) $\widetilde{K}_p^{1_c}(\mathcal{U}) = \prod_{\mathcal{U}} L_p(\mathcal{M}; \ell_1^c(\sigma))$ *and* $K_p^{1_c}(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{K}_{\widetilde{p}}^{1_c}(\mathcal{U}))}^{\|\ \|_{\widetilde{K}_p^{1_c}(\mathcal{U})}}$,

where $I_{\widetilde{p},p} : \widetilde{K}_{\widetilde{p}}^{1_c}(\mathcal{U}) \to \widetilde{K}_p^{1_c}(\mathcal{U})$ *denotes the contractive ultraproduct of the componentwise inclusion maps.*

(ii) $\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U}) = \prod_{\mathcal{U}} h_p^{1_c}(\sigma)$ *and* $\mathsf{h}_p^{1_c}(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U}))}^{\|\ \|_{\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})}}$,

where $I_{\widetilde{p},p} : \widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U}) \to \widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})$ *denotes the contractive ultraproduct of the componentwise inclusion maps.*

(iii) $\widetilde{\mathsf{h}}_p^{d}(\mathcal{U}) = \prod_{\mathcal{U}} h_p^{d}(\sigma)$ *and* $\mathsf{h}_p^{d}(\mathcal{U}) = \overline{L_2(\mathcal{M}_{\mathcal{U}}) \cap \widetilde{\mathsf{h}}_p^{d}(\mathcal{U})}^{\|\cdot\|_{\widetilde{\mathsf{h}}_p^{d}(\mathcal{U})}}$.

Remark 3.4.15. 1. Note that for all $1 \leq p < 2$ we have an isometric embedding

$$i_{\mathcal{U}} : \widetilde{\mathsf{h}}_p^{1_c} \to \widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})$$

defined by $i_{\mathcal{U}}(x) = (x)^\bullet$ for $x \in L_p(\mathcal{M})$. Hence by the definition of the regularized spaces this map sends

$$i_{\mathcal{U}} : \mathsf{h}_p^{1_c} \to \mathsf{h}_p^{1_c}(\mathcal{U}) \quad \text{isometrically.}$$

2. The same holds true for h_p^d, i.e., $\widetilde{\mathsf{h}_p^d}$ embeds isometrically into $\widetilde{\mathsf{h}}_p^d(\mathcal{U})$ and h_p^d into $\mathsf{h}_p^d(\mathcal{U})$ via the map $i_{\mathcal{U}}$.

3. Adapting the discussion of Remark 3.2.21 we see that $\mathcal{E}_{\mathcal{U}}$ is well-defined on $\mathsf{h}_p^{1_c}(\mathcal{U})$ and on $\mathsf{h}_p^d(\mathcal{U})$ for $1 \leq p < 2$.

Since the definition of $\mathsf{h}_p^{1_c}(\mathcal{U})$ is more consistent with the other regularized spaces defined previously than the definition of $\mathsf{h}_p^d(\mathcal{U})$, we will establish the analogue for the ultraproduct spaces of the Davis decomposition involving the space $h_p^{1_c}$.

For $1 \leq p < 2$, let us consider $i = (i_\sigma)^\bullet$ the ultraproduct map of the isometric inclusions

$$i_\sigma : \left\{ \begin{array}{ccc} h_p^{1_c}(\sigma) & \longrightarrow & L_p(\mathcal{M}; \ell_1^c(\sigma)) \\ x & \longmapsto & (d_t^\sigma(x))_{t\in\sigma} \end{array} \right.$$

and $\mathcal{D} = (\mathcal{D}_\sigma)^\bullet$ the ultraproduct map of the Stein projections

$$\mathcal{D}_\sigma : \left\{ \begin{array}{ccc} L_p(\mathcal{M}; \ell_1^c(\sigma)) & \longrightarrow & h_p^{1_c}(\sigma) \\ (x_t)_{t\in\sigma} & \longmapsto & \sum_{t\in\sigma} d_t^\sigma(x_t) \end{array} \right. .$$

Then $i : \widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U}) \to \widetilde{K}_p^{1_c}(\mathcal{U})$ is still isometric, and for $x \in \widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})$ we have $x = \mathcal{D}(i(x))$. By definition we see that i and $\mathcal{D}$ preserve the regularized spaces, i.e.,

$$i : \mathsf{h}_p^{1_c}(\mathcal{U}) \to K_p^{1_c}(\mathcal{U}) \quad \text{and} \quad \mathcal{D} : K_p^{1_c}(\mathcal{U}) \to \mathsf{h}_p^{1_c}(\mathcal{U}).$$

According to Lemma 3.4.5 (i), $\mathcal{D}$ is a bounded projection for $1 \leq p < 2$.

The Davis decomposition of Theorem 3.4.7 for $1 \leq p < 2$ in the discrete case applied to each partition σ immediately implies the analoguous result for the ultraproduct spaces.

Proposition 3.4.16. *Let $1 \le p < 2$. Then*

$$\widetilde{\mathcal{H}}_p^c(\mathcal{U}) = \phi_p^{1_c}(\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})) + \phi_p^c(\widetilde{\mathsf{h}}_p^c(\mathcal{U})) \quad \textit{with equivalent norms,}$$

where $\phi_p^{1_c} : \widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U}) \to \widetilde{\mathcal{H}}_p^c(\mathcal{U})$ and $\phi_p^c : \widetilde{\mathsf{h}}_p^c(\mathcal{U}) \to \widetilde{\mathcal{H}}_p^c(\mathcal{U})$ denote the (non necessarily injective) ultraproduct maps of the componentwise inclusions.

By definition, the maps $\phi_p^{1_c}$ and ϕ_p^c preserve the regularized spaces. Hence we can state the Davis decomposition for the regularized spaces.

Proposition 3.4.17. *Let $1 \le p < 2$. Then*

$$\mathcal{H}_p^c(\mathcal{U}) = \phi_p^{1_c}(\mathsf{h}_p^{1_c}(\mathcal{U})) + \phi_p^c(\mathsf{h}_p^c(\mathcal{U})) \quad \textit{with equivalent norms.}$$

Proof. The proof is similar to that of Proposition 3.2.55. Here we need the fact that for $x \in \mathcal{H}_p^c(\mathcal{U})$,

$$\|\xi\|_{\widetilde{\mathcal{H}}_p^c(\mathcal{U})} = \lim_{q \overset{>}{\to} p} \|\xi\|_{\widetilde{\mathcal{H}}_q^c(\mathcal{U})}.$$

This follows directly from Lemma 3.2.19 and the isometry $i : \mathcal{H}_p^c(\mathcal{U}) \to K_p^c(\mathcal{U})$. $\square$

3.4.4 The Davis decomposition for $\mathcal{H}_p^c$

It now remains to apply the conditional expectation $\mathcal{E}_\mathcal{U}$ to each side of Proposition 3.4.17 to deduce the decomposition of the space $\mathcal{H}_p^c$. Indeed, as for the $\mathcal{H}_p^c$ and the h_p^c-spaces we have the following boundedness of $\mathcal{E}_\mathcal{U}$.

Proposition 3.4.18. *Let $1 \le p \le 2$. Then $\mathcal{E}_\mathcal{U} : \mathsf{h}_p^{1_c}(\mathcal{U}) \to \mathsf{h}_p^{1_c}$ is a contractive projection.*

Proof. Let $x = (x_\sigma)^\bullet \in \mathsf{h}_p^{1_c}(\mathcal{U})$ be such that $\|x\|_{\mathsf{h}_p^{1_c}(\mathcal{U})} < 1$. We may assume that for all σ, $\|x\|_{h_p^{1_c}(\sigma)} < 1$. By density, it suffices to consider $x = I_{\widetilde{p},p}(y)$ for $y = (y_\sigma)^\bullet \in \widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U})$ and some $\widetilde{p} > p$. For the sake of simplicity in this proof we will forget the notation $I_{\widetilde{p},p}$, and we simply assume that $x \in \widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U})$. Note that $\mathcal{E}_\mathcal{U}(x)$ is the weak-limit of the x_σ's in $L_{\widetilde{p}}(\mathcal{M})$, and can be approximated by convex combinations in $L_{\widetilde{p}}$-norm. For a fixed partition σ_0 and $\varepsilon > 0$, we can find a sequence of positive numbers $(\alpha_m)_{m=1}^M$ such that $\sum_m \alpha_m = 1$, and partitions $\sigma^1, ..., \sigma^M$ containing σ_0 such that

$$\Big\| \mathcal{E}_\mathcal{U}(x) - \sum_m \alpha_m x_{\sigma^m} \Big\|_{\widetilde{p}} < \varepsilon$$

and

$$\|x_{\sigma^m}\|_{h_{\widetilde{p}}^{1_c}(\sigma^m)} < (1+\varepsilon)\|x\|_{\widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U})} \quad \text{for all } m = 1, \cdots, M.$$

Then we deduce that

$$\begin{aligned} \|\mathcal{E}_\mathcal{U}(x)\|_{h_{\widetilde{p}}^{1_c}(\sigma_0)} &\le \Big\| \mathcal{E}_\mathcal{U}(x) - \sum_m \alpha_m x_{\sigma^m} \Big\|_{h_{\widetilde{p}}^{1_c}(\sigma_0)} + \Big\| \sum_m \alpha_m x_{\sigma^m} \Big\|_{h_{\widetilde{p}}^{1_c}(\sigma_0)} \\ &\le 2\varepsilon|\sigma_0| + \sum_m \alpha_m \|x_{\sigma^m}\|_{h_{\widetilde{p}}^{1_c}(\sigma_0)}. \end{aligned}$$

The last inequality comes from the fact that for $1 \le p < 2, z \in L_p(\mathcal{M})$ and σ_0 a finite partition we have

$$\|z\|_{h_p^{1_c}(\sigma_0)} \le 2|\sigma_0| \|z\|_p. \tag{3.4.4}$$

Indeed, by the triangle inequality in $h_p^{1_c}(\sigma_0)$ we have $\|z\|_{h_p^{1_c}(\sigma_0)} \leq \sum_{t\in\sigma_0} \|d_t^{\sigma_0}(z)\|_{h_p^{1_c}(\sigma_0)}$. We can write $(\delta_{s,t} d_t^{\sigma_0}(z))_{s\in\sigma_0} = (b_s a_s)_{s\in\sigma_0}$ with

$$b_s = \delta_{s,t} v_t |d_t^{\sigma_0}(z)|^{p/2} \quad \text{and} \quad a_s = \delta_{s,t} |d_t^{\sigma_0}(z)|^{p/q},$$

where $d_t^{\sigma_0}(z) = v_t |d_t^{\sigma_0}(z)|$ is the polar decomposition of $d_t^{\sigma_0}(z)$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Then we obtain

$$\|d_t^{\sigma_0}(z)\|_{h_p^{1_c}(\sigma_0)} \leq \|v_t |d_t^{\sigma_0}(z)|^{p/2}\|_2 \||d_t^{\sigma_0}(z)|^{p/q}\|_q \leq \|d_t^{\sigma_0}(z)\|_p \leq 2\|z\|_p$$

and (3.4.4) follows. We get by Lemma 3.4.9

$$\|\mathcal{E}_{\mathcal{U}}(x)\|_{h_{\widetilde{p}}^{1_c}(\sigma_0)} \leq 2|\sigma_0|\varepsilon + \sum_m \alpha_m \|x_{\sigma^m}\|_{h_{\widetilde{p}}^{1_c}(\sigma^m)} \leq 2|\sigma_0|\varepsilon + (1+\varepsilon)\|x\|_{\widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U})}.$$

Sending ε to 0 implies that $\mathcal{E}_{\mathcal{U}}(x) \in \widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}$. Hence $\mathcal{E}_{\mathcal{U}}(x) \in \mathsf{h}_p^{1_c}$ and the same argument gives

$$\|\mathcal{E}_{\mathcal{U}}(x)\|_{\mathsf{h}_p^{1_c}} \leq \|x\|_{\mathsf{h}_p^{1_c}(\mathcal{U})}.$$

□

The same result holds true for the space $\mathsf{h}_p^d(\mathcal{U})$. Since we will need it later in section 3.5, let us state and prove this result here.

Lemma 3.4.19. *Let $1 \leq p < 2$. Then $\mathcal{E}_{\mathcal{U}} : \mathsf{h}_p^d(\mathcal{U}) \to \mathsf{h}_p^d$ is a bounded projection.*

Proof. By density, it suffices to consider $x = (x_\sigma)^\bullet \in L_2(\mathcal{M}_{\mathcal{U}}) \cap \widetilde{\mathsf{h}}_p^d(\mathcal{U})$ such that $\|x\|_{h_p^d(\mathcal{U})} < 1$. We may assume that for all σ, $\|x_\sigma\|_{h_p^d(\sigma)} < 1$. Then we have $\mathcal{E}_{\mathcal{U}}(x) = w\text{-}\lim_{\sigma,\mathcal{U}} x_\sigma$ in L_2. Let us fix a partition σ and $\varepsilon > 0$. We can find positive numbers $(\alpha_m)_{m=1}^M$ such that $\sum_m \alpha_m = 1$ and partitions $\sigma^1, \cdots, \sigma^M$ containing σ such that

$$\Big\|\mathcal{E}_{\mathcal{U}}(x) - \sum_m \alpha_m x_{\sigma^m}\Big\|_2 < \varepsilon.$$

Note that since σ is a finite partition, by the Hölder inequality for $\ell_p(\sigma; L_p(\mathcal{M}))$ we have for $y \in L_2(\mathcal{M})$

$$\|y\|_{h_p^d(\sigma)} = \|(d_t^\sigma(y))_{t\in\sigma}\|_{\ell_p(\sigma;L_p(\mathcal{M}))} \leq |\sigma|^{1/q}\|y\|_2,$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Hence by Lemma 3.4.9 we get

$$\begin{aligned}
\|\mathcal{E}_{\mathcal{U}}(x)\|_{h_p^d(\sigma)} &\leq \Big\|\mathcal{E}_{\mathcal{U}}(x) - \sum_m \alpha_m x_{\sigma^m}\Big\|_{h_p^d(\sigma)} + \Big\|\sum_m \alpha_m x_{\sigma^m}\Big\|_{h_p^d(\sigma)} \\
&\leq \varepsilon|\sigma|^{1/q} + \sum_m \alpha_m \|x_{\sigma^m}\|_{h_p^d(\sigma)} \leq \varepsilon|\sigma|^{1/q} + 2\sum_m \alpha_m \|x_{\sigma^m}\|_{h_p^d(\sigma^m)} \\
&\leq \varepsilon|\sigma|^{1/q} + 2.
\end{aligned}$$

Sending ε to 0, we obtain that $\|\mathcal{E}_{\mathcal{U}}(x)\|_{h_p^d(\sigma)} \leq 2$ for all σ, thus

$$\|\mathcal{E}_{\mathcal{U}}(x)\|_{\mathsf{h}_p^d} \leq 2\|x\|_{\mathsf{h}_p^d(\mathcal{U})}.$$

□

Finally, combining Proposition 3.4.17 with Propositions 3.2.38, 3.3.28 and 3.4.18 we get

Theorem 3.4.20. *Let $1 \le p < 2$. Then*

$$\mathcal{H}_p^c = \mathsf{h}_p^{1_c} + \mathsf{h}_p^c \quad \textit{with equivalent norms.}$$

Proof. We apply $\mathcal{E}_\mathcal{U}$ to Proposition 3.4.17. It suffices to observe that the following diagrams are commuting

$$\begin{array}{ccc} \mathsf{h}_p^{1_c}(\mathcal{U}) & \xrightarrow{\phi_p^{1_c}} & \mathcal{H}_p^c(\mathcal{U}) \\ \downarrow{\scriptstyle \mathcal{E}_\mathcal{U}} & & \downarrow{\scriptstyle \mathcal{E}_\mathcal{U}} \\ \mathsf{h}_p^{1_c} & \xrightarrow[id]{} & \mathcal{H}_p^c \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathsf{h}_p^{c}(\mathcal{U}) & \xrightarrow{\phi_p^{c}} & \mathcal{H}_p^c(\mathcal{U}) \\ \downarrow{\scriptstyle \mathcal{E}_\mathcal{U}} & & \downarrow{\scriptstyle \mathcal{E}_\mathcal{U}} \\ \mathsf{h}_p^{c} & \xrightarrow[id]{} & \mathcal{H}_p^c \end{array}.$$

□

We end this section with the analogue of the usual Davis decomposition stated in Theorem 3.4.1. We first establish a density result for the regularized space $\mathsf{h}_p^{1_c}(\mathcal{U})$.

Lemma 3.4.21. *Let $1 \le p < 2$. Then $L_2(\mathcal{M}_\mathcal{U}) \cap \mathsf{h}_p^{1_c}(\mathcal{U})$ is dense in $\mathsf{h}_p^{1_c}(\mathcal{U})$.*

Proof. Let $\widetilde{p} > p$ and $x = (x_\sigma)^\bullet \in \widetilde{h}_{\widetilde{p}}^{1_c}(\mathcal{U})$ be such that $\|x\|_{\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})} < 1$ and $\|x\|_{\widetilde{\mathsf{h}}_{\widetilde{p}}^{1_c}(\mathcal{U})} < C$. We assume that for all σ, $\|x_\sigma\|_{h_p^{1_c}(\sigma)} < 1$ and $\|x_\sigma\|_{h_{\widetilde{p}}^{1_c}(\sigma)} < C$. Let $\widetilde{q} > q$ be such that $\frac{1}{\widetilde{p}} = \frac{1}{2} + \frac{1}{\widetilde{q}}$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Then by Lemma 3.4.2 for each σ we can decompose $d_t^\sigma(x_\sigma) = b_\sigma(t)a_\sigma(t)$ with

$$\Big(\sum_{t\in\sigma} \|b_\sigma(t)\|_2^2\Big)^{1/2} \Big\|\Big(\sum_{t\in\sigma} |a_\sigma(t)|^2\Big)^{1/2}\Big\|_q < 1 \quad \text{and} \quad \Big(\sum_{t\in\sigma} \|b_\sigma(t)\|_2^2\Big)^{1/2} \Big\|\Big(\sum_{t\in\sigma} |a_\sigma(t)|^2\Big)^{1/2}\Big\|_{\widetilde{q}} < C.$$

We set

$$\alpha = \Big(\sum_{t\in\sigma} e_{t,0} \otimes a_\sigma(t)\Big)^\bullet \in \widetilde{K_{\widetilde{q}}^c}(\mathcal{U}).$$

By Corollary 3.2.18 (iii) we see that α is in the L_p $\mathcal{M}_\mathcal{U}$-module $K_q^c(\mathcal{U})$. Hence Lemma 3.2.17 implies that for $\varepsilon > 0$ we can find an element $\widetilde{\alpha} \in K_\infty^c(\mathcal{U})$ such that $\|\alpha - \widetilde{\alpha}\|_q \le \varepsilon$. Moreover, we may assume that $\widetilde{\alpha} = \Big(\sum_{t\in\sigma} e_{t,0} \otimes \widetilde{a}_\sigma(t)\Big)^\bullet \in \prod_\mathcal{U} L_\infty(\mathcal{M}; \ell_2^c(\sigma))$. Indeed, by Lemma 3.1.3, the unit ball of $\prod_\mathcal{U} L_\infty(\mathcal{M}; \ell_2^c(\sigma))$ is weak*-dense in the unit ball of $\Big(\prod_\mathcal{U} L_1(\mathcal{M}; \ell_2^c(\sigma))\Big)^* = (\widetilde{K_1^c}(\mathcal{U}))^*$. Hence $\widetilde{\alpha} \in \widetilde{K_\infty^c}(\mathcal{U}) = (\widetilde{K_1^c}(\mathcal{U}))^*$ and we can find an uniformly bounded sequence $(\widetilde{\alpha}_\lambda)_\lambda$ in $\prod_\mathcal{U} L_\infty(\mathcal{M}; \ell_2^c(\sigma))$ which weak*-converges to $\widetilde{\alpha}$. Then we easily see that $\|\widetilde{\alpha} - \widetilde{\alpha}_\lambda\|_2 \to 0$. For $2 < q < \infty$ we write

$$\|\widetilde{\alpha} - \widetilde{\alpha}_\lambda\|_{K_q^c(\mathcal{U})} \le \|\widetilde{\alpha} - \widetilde{\alpha}_\lambda\|_{K_2^c(\mathcal{U})}^{2/q} \|\widetilde{\alpha} - \widetilde{\alpha}_\lambda\|_{K_\infty^c(\mathcal{U})}^{1-2/q}.$$

Since the sequence $(\widetilde{\alpha}_\lambda)_\lambda$ is uniformly bounded in $K_\infty^c(\mathcal{U})$, this shows that $\widetilde{\alpha}_\lambda \to \widetilde{\alpha}$ in $K_q^c(\mathcal{U})$. Now we set

$$\widetilde{x} = (\widetilde{x}_\sigma)^\bullet \quad \text{with} \quad \widetilde{x}_\sigma = (b_\sigma(t)\widetilde{a}_\sigma(t))_{t\in\sigma}.$$

Then $\widetilde{x} \in K_p^{1_c}(\mathcal{U})$, hence $y = \mathcal{D}(\widetilde{x}) \in \mathsf{h}_p^{1_c}(\mathcal{U})$ and by the Hölder inequality we have $y \in L_2(\mathcal{M}_\mathcal{U})$. Moreover, Lemma 3.4.5 (i) implies

$$\begin{aligned} \|x - y\|_{\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})} &= \|\mathcal{D}(i(x)) - \mathcal{D}(\widetilde{x})\|_{\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})} \le 2\delta'^{1/2}_{q/2} \|i(x) - \widetilde{x}\|_{\widetilde{K}_p^{1_c}(\mathcal{U})} \\ &\le 2\delta'^{1/2}_{q/2} \lim_{\sigma,\mathcal{U}} \Big(\sum_{t\in\sigma} \|b_\sigma(t)\|_2^2\Big)^{1/2} \Big\|\Big(\sum_{t\in\sigma} |a_\sigma(t) - \widetilde{a}_\sigma(t)|^2\Big)^{1/2}\Big\|_q < 2\delta'^{1/2}_{q/2}\varepsilon. \end{aligned}$$

□

Combining this with Proposition 3.4.18 we get

Corollary 3.4.22. *Let $1 \leq p < 2$. Then $L_2(\mathcal{M}) \cap \mathsf{h}_p^{1_c}$ is dense in $\mathsf{h}_p^{1_c}$.*

We can now define by density a contractive map from $\mathsf{h}_p^{1_c}$ to h_p^d, which is injective for $\mathsf{h}_p^{1_c}$ and h_p^d are subspaces of $L_p(\mathcal{M})$.

Corollary 3.4.23. *Let $1 \leq p < 2$. Then we have a contractive inclusion*

$$\mathsf{h}_p^{1_c} \subset \mathsf{h}_p^d.$$

We deduce from Theorem 3.4.20 and Remark 3.4.13 (2) the desired Davis decomposition.

Theorem 3.4.24. *Let $1 \leq p < 2$. Then*

$$\mathcal{H}_p^c = \mathsf{h}_p^d + \mathsf{h}_p^c \quad \textit{with equivalent norms.}$$

3.4.5 The Burkholder inequalities for $1 < p < 2$

Combining Theorem 3.4.24 with Theorem 3.2.56 we deduce the analogue of the noncommutative Burkholder inequalities for $1 < p < 2$.

Definition 3.4.25. *Let $1 \leq p < 2$. We define*

$$\mathsf{h}_p = \mathsf{h}_p^d + \mathsf{h}_p^c + \mathsf{h}_p^r,$$

where the sum is taken in $L_p(\mathcal{M})$.

Theorem 3.4.26. *Let $1 < p < 2$. Then*

$$L_p(\mathcal{M}) = \mathsf{h}_p \quad \textit{with equivalent norms.}$$

The Burkholder inequalities for $2 < p < \infty$ will be discussed in the next section.

3.5 Stronger decompositions for $1 \leq p < 2$ and the Burkolder inequalities for $2 < p < \infty$

The aim of this section is to establish the Burkolder inequalities and the analogue of Theorem 3.4.1 for $2 < p < \infty$. In particular we will define and study the diagonal space h_p^d in this case, more precisely we will discuss its intersection with the conditioned Hardy spaces h_p^c and h_p^r. We will follow a dual approach, and start by considering decompositions for $1 < p < 2$. We introduce another construction for the sum of Banach spaces, which is stronger in some sense than the usual sum. After showing that the decompositions we consider also hold true for this construction, we will study their dual versions and obtain new results for $2 < p < \infty$. We end the section with a discussion on the case $p = 1$, and establish a stronger Fefferman-Stein duality for $\mathcal{H}_1^c$.

3.5.1 Sums of Banach spaces

There are plenty opportunities to consider sums of Banach spaces and we will consider two competing constructions. Let X and Y be two Banach spaces both embedded into a Banach space A_1, i.e., the inclusion maps $X \subset A_1$ and $Y \subset A_1$ are continuous and injective. In interpolation theory one considers the sum

$$X + Y = \{z \in A_1 : \exists\, x \in X, y \in Y \text{ such that } z = x + y\}$$

equipped with the norm

$$\|z\|_{X+Y} = \inf_{z=x+y} \|x\|_X + \|y\|_Y.$$

Note that $X + Y$ is complete and isomorphic to the quotient space $X \oplus_1 Y/L$, where

$$L = \ker \phi = \{(x, -x) \in X \oplus_1 Y \;:\; x \in X \cap Y\}$$

and

$$\phi : \begin{cases} X \oplus_1 Y & \longrightarrow & A_1 \\ (x, y) & \longmapsto & x + y \end{cases}.$$

The second method we will consider depends on a fourth space A_0, which is also injectively embedded into A_1. We assume that

$$A_0 \cap X \text{ is dense in } X \text{ and } A_0 \cap Y \text{ is dense in } Y. \tag{3.5.1}$$

We will define the A_0 sum

$$X \boxplus_{A_0} Y$$

as the completion of the quotient

$$((A_0 \cap X) \oplus_1 (A_0 \cap Y))/L_0,$$

where

$$L_0 = \ker(\phi_{|(A_0\cap X)\oplus_1(A_0\cap Y)}) = L\cap((A_0\cap X)\oplus_1(A_0\cap Y)) = \{(x, -x) \in X\oplus_1 Y \;:\; x \in A_0\cap X\cap Y\}.$$

In our context we will always consider $A_0 = L_2(\mathcal{M})$, and simply denote $X \boxplus Y$.

Let us state the following basic fact.

Lemma 3.5.1. *Let A_0, X, Y, A_1 be four Banach spaces as above. Then there exists a surjective quotient map $q : X \boxplus Y \to X + Y$.*

Proof. Since $L_0 \subset L$, we have a contractive map

$$q : \begin{cases} ((A_0 \cap X) \oplus_1 (A_0 \cap Y))/L_0 & \longrightarrow & X + Y \\ (x, y) + L_0 & \longmapsto & (x, y) + L \end{cases}.$$

Since $X + Y$ is a Banach space, we deduce that q uniquely extends to the completion $X \boxplus Y$, still denoted by q. Let us show that q is a quotient map. Let $z \in X + Y$ be of norm < 1. We can find $x \in X$ and $y \in Y$ such that $z = x + y$ and $\|x\|_X = \lambda, \|y\|_Y = \mu$ with $\lambda + \mu < 1$. Since $A_0 \cap X$ is dense in X, we can find a sequence $(x_n)_n$ in $A_0 \cap X$ such that the series is absolutely converging and

$$\sum_n \|x_n\|_X \leq \lambda + \frac{1 - (\lambda + \mu)}{4} \quad , \quad x = \sum_n x_n \text{ in } X.$$

Similarly, there exists a sequence $(y_n)_n$ in $A_0 \cap Y$ such that the series is absolutely converging and

$$\sum_n \|y_n\|_Y \le \mu + \frac{1-(\lambda+\mu)}{4} \quad , \quad y = \sum_n y_n \text{ in } Y.$$

Then $z_n = (x_n, y_n) + L_0 \in X \boxplus Y$ for all n, and

$$\sum_n \|z_n\|_{X \boxplus Y} \le \sum_n \|x_n\|_X + \|y_n\|_Y \le \frac{1+\lambda+\mu}{2} < 1.$$

Hence the series $(z_n)_n$ is absolutely converging in $X \boxplus Y$ and we have

$$q\Big(\sum_n z_n\Big) = z.$$

This ends the proof. $\square$

Remark 3.5.2. In the situations considered later, we will always have $A_0 \subset Y$. In this case, $X \boxplus Y$ is isometrically isomorphic to the completion of A_0 with respect to the norm

$$|||z|||_{X \boxplus Y} = \inf_{z=x+y, x \in A_0 \cap X, y \in A_0} \|x\|_X + \|y\|_Y.$$

Indeed, in this case the restriction ϕ_0 of the map ϕ to $(A_0 \cap X) \oplus_1 (A_0 \cap Y) = (A_0 \cap X) \oplus_1 A_0$

$$\phi_0 : (A_0 \cap X) \oplus_1 A_0 \to A_0$$

is surjective. This comes from the fact that for $a \in A_0$, we may write $a = \phi_0(0, a)$. Hence the quotient space $\big((A_0 \cap X) \oplus_1 (A_0 \cap Y)\big)/L_0$ is isomorphic to A_0. It remains to see that the norms $\|\cdot\|_{X \boxplus Y}$ and $|||\cdot|||_{X \boxplus Y}$ coincide on A_0. This follows directly from the fact that for $a \in A_0$,

$$\{(x,y) \in (A_0 \cap X) \oplus_1 A_0 \ : \ (x,y) + L_0 = (0,a) + L_0\} = \{(x,y) \in (A_0 \cap X) \times A_0 \ : \ a = x + y\}.$$

Observe that by using this characterization, it is easy to show that for a Banach space $A_0 \subset Z \subset A_1$ such that A_0 is dense in Z, we have the associativity relation

$$(X \boxplus Z) \boxplus Y = X \boxplus (Z \boxplus Y).$$

The two sums coincide in the following cases.

Lemma 3.5.3. *Let A_0, X, Y, A_1 be four Banach spaces as above. Then the following assertions are equivalent.*

(i) $X + Y = X \boxplus Y$ isometrically;

(ii) $X \boxplus Y$ embeds injectively into A_1;

(iii) $A_0 \cap X \cap Y$ is dense in $X \cap Y$.

Proof. By Lemma 3.5.1, we see that the two sums coincide isometrically if and only if the quotient map q is injective. Let us consider the following commuting diagram

$$\begin{tikzcd} X \boxplus Y \arrow[r, "q"] \arrow[dr, "f"'] & X + Y \arrow[d, hook, "id"] \\ & A_1 \end{tikzcd} .$$

It is clear that q is injective if and only if f is injective. This proves (i)$\Leftrightarrow$(ii). For the equivalence (i)$\Leftrightarrow$(iii), it suffices to observe that the density assumption (3.5.1) yields

$$X \boxplus Y = \overline{((A_0 \cap X) \oplus_1 (A_0 \cap Y))/L_0} = X \oplus_1 Y/\overline{L_0}.$$

Hence

$$X + Y = X \boxplus Y \Leftrightarrow L = \overline{L_0} \Leftrightarrow L_0 \text{ is dense in } L \Leftrightarrow A_0 \cap X \cap Y \text{ is dense in } X \cap Y.$$

□

We will be also concerned with the corresponding dual spaces. In interpolation theory it is well-known that the dual of a sum is an intersection. However, it is usually assumed that the intersection is dense in each space, a condition which might no longer be satisfied in our applications.

Lemma 3.5.4. *Let A_0, X, Y, A_1 be four Banach spaces as above.*

(i) The dual space of $X + Y$ consists of pairs $(x^*, y^*) \in X^* \oplus_\infty Y^*$ such that

$$x^*|_{X\cap Y} = y^*|_{X\cap Y}. \tag{3.5.2}$$

We equip the dual space $(X + Y)^*$ with the norm

$$\|\phi\| = \max\{\|x^*\|_{X^*}, \|y^*\|_{Y^*}\} \quad \textit{where} \quad \phi((x, y) + L) = x^*(x) + y^*(y).$$

This space is usually denoted by $X^* \cap Y^*$.

(ii) The dual space of $X \boxplus Y$ consists of pairs $(x^*, y^*) \in X^* \oplus_\infty Y^*$ such that

$$x^*|_{A_0\cap X\cap Y} = y^*|_{A_0\cap X\cap Y}. \tag{3.5.3}$$

We equip the dual space $(X \boxplus Y)^*$ with the norm

$$\|\phi\| = \max\{\|x^*\|_{X^*}, \|y^*\|_{Y^*}\} \quad \textit{where} \quad \phi((x, y) + L_0) = x^*(x) + y^*(y).$$

This space will be denoted by $X^* \Cap Y^*$.

Proof. The proofs of (i) and (ii) are similar. Let ϕ be a functional on $X + Y$. Since $X + Y$ is a quotient space of $X \oplus_1 Y$, ϕ is given by

$$\phi((x, y) + L) = x^*(x) + y^*(y) \quad \text{for} \ \ (x, y) \in X \oplus_1 Y$$

where $(x^*, y^*) \in (X \oplus_1 Y)^* = X^* \oplus_\infty Y^*$ satisfies

$$x^*(x) + y^*(y) = 0 \quad \text{for all} \ \ (x, y) \in L.$$

This is equivalent to the condition (3.5.2). For (ii), by density a functional ϕ on $X \boxplus Y$ is determined by a functional on the quotient space $((A_0 \cap X) \oplus_1 (A_0 \cap Y))/L_0$. Hence ϕ is given by

$$\phi((x, y) + L_0) = x^*(x) + y^*(y) \quad \text{for} \ \ (x, y) \in (A_0 \cap X) \oplus_1 (A_0 \cap Y)$$

where $(x^*, y^*) \in X^* \oplus_\infty Y^*$ satisfies

$$x^*(x) + y^*(y) = 0 \quad \text{for all} \ \ (x, y) \in L_0.$$

This is equivalent to the condition (3.5.3). □

3.5.2 The Burkholder-Gundy/Burkholder/Davis decompositions for $1 < p < 2$ in the discrete case

Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration. In this subsection we recall Randrianantoanina's results on Burkholder-Gundy and Burkholder decompositions at the L_2-level. We could translate these results by using the notations introduced in the previous subsection. We see that he actually proved the Burkholder-Gundy and Burkholder decompositions for the sum $\boxplus$.

Let us first recall the stronger Burkholder-Gundy and Burkholder inequalities for $1 < p < 2$ proved in [40] and [41] respectively.

Theorem 3.5.5. *Let $1 < p < 2$ and $x \in L_2(\mathcal{M})$. Then*

1. *There exist $a, b \in L_2(\mathcal{M})$ such that*
   (i) $x = a + b$,
   (ii) $\|a\|_{H_p^c} + \|b\|_{H_p^r} \leq C(p)\|x\|_p$,
   (iii) $\max\{\|a\|_2, \|b\|_2\} \leq f(p, \|x\|_p, \|x\|_2)$.
2. *There exist $a, b, c \in L_2(\mathcal{M})$ such that*
   (i) $x = a + b + c$,
   (ii) $\|a\|_{h_p^d} + \|b\|_{h_p^c} + \|c\|_{h_p^r} \leq C(p)\|x\|_p$,
   (iii) $\max\{\|a\|_2, \|b\|_2, \|c\|_2\} \leq f(p, \|x\|_p, \|x\|_2)$.

Here $C(p) \leq C(p-1)^{-1}$ as $p \to 1$.

Proof. The proofs of (1) and (2) are similar. Let us recall Randrianantoanina's arguments for the Burkholder decomposition (2). We will derive the estimate of the L_2-norms (iii) from his construction. The main tool is the real interpolation, more precisely the J-method, to deduce this decomposition from a weak type $(1,1)$-inequality. We refer to [2] for details on interpolation. Let $x \in L_2(\mathcal{M})$ and $1 < p < 2$. Let $0 < \theta < 1$ be such that $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$. We know that $L_p(\mathcal{M}) = [L_1(\mathcal{M}), L_2(\mathcal{M})]_{\theta,p;J}$, hence we may write

$$x = \sum_{\nu\in\mathbb{Z}} u_\nu \tag{3.5.4}$$

where

$$\Big(\sum_{\nu\in\mathbb{Z}} (2^{-\nu\theta}\max\{\|u_\nu\|_1, 2^\nu\|u_\nu\|_2\})^p\Big)^{1/p} \leq C(p)\|x\|_p. \tag{3.5.5}$$

We claim that we may in addition suppose that

$$\sum_{\nu\in\mathbb{Z}} \|u_\nu\|_2 \leq f(p, \|x\|_p, \|x\|_2). \tag{3.5.6}$$

For each $\nu \in \mathbb{Z}$ we set

$$e_\nu = \mathbb{1}(\mu_{4^\nu}(x) < |x| \leq \mu_{4^{\nu-1}}(x)),$$

where for $t > 0$, $\mu_t(x)$ denote the generalized singular numbers of x. We refer to [9] for details on these generalized numbers. Since $\mu_t(x) \to \|x\|$ as $t \to 0$ and $\mu_t(x) \to 0$ as $t \to \infty$, we see that $\sum_{\nu\in\mathbb{Z}} e_\nu = s(|x|)$, where $s(|x|)$ denotes the support projection of x. Hence we can write

$$x = \sum_{\nu\in\mathbb{Z}} x e_\nu. \tag{3.5.7}$$

Let us first show that the sequence $u_\nu = xe_\nu$ satisfy (3.5.5) with $C(p) = \left(\frac{16}{3}\right)^{1/p}$. Note that by the definition of $\mu_t(x)$ we have for all ν

$$\tau(e_\nu) \leq \tau(\mathbb{1}(\mu_{4^\nu}(x) < |x|)) \leq 4^\nu. \tag{3.5.8}$$

On the other hand, since $\mu_t(x)$ is decreasing we have

$$\begin{aligned}\|x\|_p^p &= \int_0^\infty \mu_t(x)^p dt = \sum_{\nu\in\mathbb{Z}} \int_{4^{\nu-2}}^{4^{\nu-1}} \mu_t(x)^p dt \\ &\geq \sum_{\nu\in\mathbb{Z}} (4^{\nu-1} - 4^{\nu-2})\mu_{4^{\nu-1}}(x)^p = \sum_{\nu\in\mathbb{Z}} 3.4^{\nu-2}\mu_{4^{\nu-1}}(x)^p.\end{aligned}$$

By (3.5.8) we have

$$\|xe_\nu\|_1 = \tau\big(|x|\mathbb{1}(\mu_{4^\nu}(x) < |x| \leq \mu_{4^{\nu-1}}(x))\big) \leq \mu_{4^{\nu-1}}(x)\tau(e_\nu) \leq \mu_{4^{\nu-1}}(x)4^\nu.$$

Using $p(2-\theta) = 2$ we get

$$\sum_{\nu\in\mathbb{Z}} (2^{-\nu\theta}\|xe_\nu\|_1)^p \leq \sum_{\nu\in\mathbb{Z}} 2^{\nu p(2-\theta)}\mu_{4^{\nu-1}}(x)^p = \sum_{\nu\in\mathbb{Z}} 4^\nu \mu_{4^{\nu-1}}(x)^p \leq \frac{16}{3}\|x\|_p^p.$$

We also have

$$\|xe_\nu\|_2 = \tau\big(|x|^2\mathbb{1}(\mu_{4^\nu}(x) < |x| \leq \mu_{4^{\nu-1}}(x))\big)^{1/2} \leq \mu_{4^{\nu-1}}(x)\tau(e_\nu)^{1/2} \leq \mu_{4^{\nu-1}}(x)2^\nu,$$

hence

$$\sum_{\nu\in\mathbb{Z}} (2^{\nu(1-\theta)}\|xe_\nu\|_2)^p \leq \sum_{\nu\in\mathbb{Z}} 2^{\nu p(2-\theta)}\mu_{4^{\nu-1}}(x)^p \leq \frac{16}{3}\|x\|_p^p.$$

Let us now consider $\nu_0 \in \mathbb{Z}$. Then, to obtain (3.5.6), we replace decomposition (3.5.7) by

$$x = \sum_{\nu\geq\nu_0} x\widetilde{e}_\nu,$$

where $\widetilde{e}_\nu = e_\nu$ for $\nu > \nu_0$ and $\widetilde{e}_{\nu_0} = \sum_{\nu\leq\nu_0} e_\nu = \mathbb{1}(\mu_{4^{\nu_0}}(x) < |x|)$. For a good choice of ν_0, we can show that this decomposition still satisfy (3.5.5) with $C(p) = \left(\frac{19}{3}\right)^{1/p}$. Note that

$$2^{-\nu_0\theta}\|x\widetilde{e}_{\nu_0}\|_1 = 2^{-\nu_0\theta}\|x\mathbb{1}(\mu_{4^{\nu_0}}(x) < |x|)\|_1 \leq 2^{-\nu_0\theta}\|x\|_2\tau(\mathbb{1}(\mu_{4^{\nu_0}}(x) < |x|))^{1/2} \leq 2^{\nu_0(1-\theta)}\|x\|_2$$

and

$$2^{\nu_0(1-\theta)}\|x\widetilde{e}_{\nu_0}\|_2 \leq 2^{\nu_0(1-\theta)}\|x\|_2.$$

We can find $\nu_0 = \nu_0(p, \|x\|_p, \|x\|_2)$ such that

$$2^{\nu_0(1-\theta)}\|x\|_2 \leq \|x\|_p \Leftrightarrow \nu_0 \leq (1-\theta)^{-1}\ln\Big(\frac{1}{2}\Big)\ln\Big(\frac{\|x\|_p}{\|x\|_2}\Big).$$

We then obtain

$$\Big(\sum_{\nu\geq\nu_0} (2^{-\nu\theta}\max\{\|x\widetilde{e}_\nu\|_1, 2^\nu\|x\widetilde{e}_\nu\|_2\})^p\Big)^{1/p} \leq \Big(\frac{19}{3}\Big)^{1/p}\|x\|_p.$$

The inequality (3.5.6) follows from the Hölder inequality

$$\sum_{\nu\geq\nu_0} \|x\widetilde{e}_\nu\|_2 \leq \Big(\sum_{\nu\geq\nu_0} (2^{\nu(1-\theta)}\|x\widetilde{e}_\nu\|_2)^p\Big)^{1/p}\Big(\sum_{\nu\geq\nu_0} 2^{-\nu(1-\theta)p'}\Big)^{1/p'} \leq f(p, \|x\|_p, \|x\|_2),$$

where
$$f(p, \|x\|_p, \|x\|_2) = \frac{2^{-\nu_0(1-\theta)}}{(1-2^{-(1-\theta)p'})^{1/p'}} \Big(\frac{19}{3}\Big)^{1/p} \|x\|_p.$$

Now we apply Randrianantoanina's decomposition to the sequence $(u_\nu)_\nu$ satisfying (3.5.4), (3.5.5) and (3.5.6). For each $\nu \in \mathbb{Z}$, by Theorem 3.1 of [41], we may find an absolute constant $K > 0$ and three adapted sequences $a^{(\nu)}, b^{(\nu)}$ and $c^{(\nu)}$ such that
$$d_n(u_\nu) = a_n^{(\nu)} + b_n^{(\nu)} + c_n^{(\nu)}, \quad \forall\, n \geq 0$$
and
$$\|a^{(\nu)}\|_{L_2(\mathcal{M};\ell_2^c)} + \|b^{(\nu)}\|_{L_2(\mathcal{M};\ell_2^c)} + \|c^{(\nu)}\|_{L_2(\mathcal{M};\ell_2^r)} \leq K\|u_\nu\|_2,$$
$$\Big\|\sum_{n\geq 0} e_{n,n}\otimes a_n^{(\nu)}\Big\|_{L_{1,\infty}(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M})} + \Big\|\Big(\sum_{n\geq 0}\mathcal{E}_{n-1}|b_n^{(\nu)}|^2\Big)^{1/2}\Big\|_{1,\infty} + \Big\|\Big(\sum_{n\geq 0}\mathcal{E}_{n-1}|(c_n^{(\nu)})^*|^2\Big)^{1/2}\Big\|_{1,\infty} \leq K\|u_\nu\|_1.$$
Recall that $\|x\|_{1,\infty} = \sup_{t>0} t\mu_t(x)$. Then we set
$$a_n = \sum_{\nu\in\mathbb{Z}} a_n^{(\nu)}, \quad b_n = \sum_{\nu\in\mathbb{Z}} b_n^{(\nu)} \quad \text{and} \quad c_n = \sum_{\nu\in\mathbb{Z}} c_n^{(\nu)},$$
and obtain three adapted sequences $a = (a_n)_n, b = (b_n)_n$ and $c = (c_n)_n$. Using the following interpolation result of noncommutative L_p-spaces associated to a semifinite von Neumann algebra $\mathcal{N}$
$$L_p(\mathcal{N}) = [L_{1,\infty}(\mathcal{N}), L_2(\mathcal{N})]_{\theta,p;J},$$
and (3.5.5) we can show that
$$\Big(\sum_{n\geq 0}\|a_n\|_p^p\Big)^{1/p} + \Big\|\Big(\sum_{n\geq 0}\mathcal{E}_{n-1}|b_n|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{n\geq 0}\mathcal{E}_{n-1}|c_n^*|^2\Big)^{1/2}\Big\|_p \leq C(p-1)^{-1}\|x\|_p.$$
Applying the Stein projection $\mathcal{D}$ to the sequences a, b and c we obtain three martingales. We set
$$a' = \mathcal{D}(a), \quad b' = \mathcal{D}(b) \quad \text{and} \quad c' = \mathcal{D}(c).$$
Then we have
$$d_n(x) = d_n(a') + d_n(b') + d_n(c') \quad \forall\, n \geq 0.$$
Moreover, since any conditional expectation $\mathcal{E}$ is a contractive projection in $L_p(\mathcal{M})$ and satisfies $\mathcal{E}(y)^*\mathcal{E}(y) \leq \mathcal{E}(y^*y)$, we get
$$\|a'\|_{h_p^d} + \|b'\|_{h_p^c} + \|c'\|_{h_p^r} \leq C'(p-1)^{-1}\|x\|_p.$$
It remains to prove the L_2-estimate (iii). This comes from (3.5.6) as follows
$$\|a'\|_2 = \|\mathcal{D}(a)\|_2 \leq 2\|a\|_2 \leq 2\sum_{\nu\in\mathbb{Z}}\|a^{(\nu)}\|_{L_2(\mathcal{M};\ell_2^c)} \leq 2K\sum_{\nu\in\mathbb{Z}}\|u_\nu\|_2 \leq 2Kf(p, \|x\|_p, \|x\|_2).$$
The estimates for b' and c' are similar. □

We can derive a column version of Theorem 3.5.5 (2). This is the following version of the Davis decomposition at the L_2-level.

Corollary 3.5.6. *Let $(\mathcal{M}_n)_{n=0}^m$ be a finite filtration of $\mathcal{M}$. Let $1 < p < 2$ and $x \in L_2(\mathcal{M})$. Then there exist $a, b \in L_2(\mathcal{M})$ such that* *(i)* $x = a + b$,

(ii) $\|a\|_{h_p^d} + \|b\|_{h_p^c} \leq C(p)\|x\|_{H_p^c}$,

(iii) $\max\{\|a\|_2, \|b\|_2\} \leq f(p, \|x\|_{H_p^c}, \|x\|_2)$,

where $C(p) \leq C(p-1)^{-1}$ *as* $p \to 1$.

Proof. We apply Theorem 3.5.5 (2) to the element

$$y = \sum_{n=0}^{m} e_{n,0} \otimes d_n(x).$$

Here we consider the finite von Neumann algebra $\mathcal{N} = \mathcal{B}(\ell_2^{m+1})\overline{\otimes}\mathcal{M}$ equipped with the filtration $\mathcal{N}_n = \mathcal{B}(\ell_2^{m+1})\overline{\otimes}\mathcal{M}_n$. We have to be careful with the trace we consider on $\mathcal{N}$. The natural trace on $\mathcal{N}$ is $\mathrm{tr}_{\mathcal{N}} = \mathrm{tr} \otimes \tau$, where tr denotes the usual trace on $\mathcal{B}(\ell_2^{m+1})$. This trace is finite, but not normalized. Since Theorem 3.1 of [41] have been proved for a normalized trace, we will also need to consider the normalized trace $\tau_{\mathcal{N}} = \frac{\mathrm{tr}}{m+1} \otimes \tau$. Observe that

$$\|y\|_{L_2(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = \|x\|_2 \quad \text{and} \quad \|y\|_{L_p(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = \|x\|_{H_p^c}.$$

As in the proof of Theorem 3.5.5, we can find a sequence $(u_\nu)_\nu$ such that $y = \sum_{\nu\in\mathbb{Z}} u_\nu$ with

$$\Big(\sum_{\nu\in\mathbb{Z}} (2^{-\nu\theta} \max\{\|u_\nu\|_{L_1(\mathcal{N},\mathrm{tr}_{\mathcal{N}})}, 2^\nu \|u_\nu\|_{L_2(\mathcal{N},\mathrm{tr}_{\mathcal{N}})}\})^p\Big)^{1/p} \leq C(p)\|y\|_{L_p(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = C(p)\|x\|_{H_p^c} \tag{3.5.9}$$

and

$$\sum_{\nu\in\mathbb{Z}} \|u_\nu\|_{L_2(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} \leq f(p, \|y\|_{L_p(\mathcal{N},\mathrm{tr}_{\mathcal{N}})}, \|y\|_{L_2(\mathcal{N},\mathrm{tr}_{\mathcal{N}})}) = f(p, \|x\|_{H_p^c}, \|x\|_2). \tag{3.5.10}$$

Applying Theorem 3.1 of [41] in $(\mathcal{N}, \tau_{\mathcal{N}})$ for each $\nu \in \mathbb{Z}$, we may find an absolute constant $K > 0$ and three adapted sequences $a^{(\nu)}, b^{(\nu)}$ and $c^{(\nu)}$ such that

$$d_n(u_\nu) = a_n^{(\nu)} + b_n^{(\nu)} + c_n^{(\nu)}, \quad \forall\, n \geq 0$$

and

$$\|a^{(\nu)}\|_{L_2(\mathcal{N},\tau_{\mathcal{N}};\ell_2^c)} + \|b^{(\nu)}\|_{L_2(\mathcal{N},\tau_{\mathcal{N}};\ell_2^c)} + \|c^{(\nu)}\|_{L_2(\mathcal{N},\tau_{\mathcal{N}};\ell_2^r)} \leq K\|u_\nu\|_{L_2(\mathcal{N},\tau_{\mathcal{N}})}, \tag{3.5.11}$$

$$\begin{aligned} \Big\|\sum_{n\geq 0} e_{n,n} \otimes a_n^{(\nu)}\Big\|_{L_{1,\infty}(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{N},\mathrm{tr}\otimes\tau_{\mathcal{N}})} &+ \Big\|\Big(\sum_{n\geq 0} \mathcal{E}_{n-1}|b_n^{(\nu)}|^2\Big)^{1/2}\Big\|_{L_{1,\infty}(\mathcal{N},\tau_{\mathcal{N}})} \\ + \Big\|\Big(\sum_{n\geq 0} \mathcal{E}_{n-1}|(c_n^{(\nu)})^*|^2\Big)^{1/2}\Big\|_{L_{1,\infty}(\mathcal{N},\tau_{\mathcal{N}})} &\leq K\|u_\nu\|_{L_1(\mathcal{N},\tau_{\mathcal{N}})}. \end{aligned} \tag{3.5.12}$$

We would like to obtain the same estimates with respect to the trace $\mathrm{tr}_{\mathcal{N}}$ to use the interpolation argument and (3.5.9). Note that for $z \in L_1(\mathcal{N})$, we have

$$\|z\|_{L_1(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = (m+1)\|z\|_{L_1(\mathcal{N},\tau_{\mathcal{N}})} \quad , \quad \|z\|_{L_{1,\infty}(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = (m+1)\|z\|_{L_{1,\infty}(\mathcal{N},\tau_{\mathcal{N}})}$$

and for $z \in L_2(\mathcal{N})$ we have

$$\|z\|_{L_2(\mathcal{N},\mathrm{tr}_{\mathcal{N}})} = \sqrt{m+1}\|z\|_{L_2(\mathcal{N},\tau_{\mathcal{N}})}.$$

Hence multiplying (3.5.11) and (3.5.12) by $\sqrt{m+1}$ and $(m+1)$ respectively, we get the same estimates with respect to the trace $\mathrm{tr}_{\mathcal{N}}$. Thus we may control the J-functionals for $a^{(\nu)}, b^{(\nu)}$ and $c^{(\nu)}$ in $(L_{1,\infty}(\mathcal{N}, \mathrm{tr}_{\mathcal{N}}), L_2(\mathcal{N}, \mathrm{tr}_{\mathcal{N}}))$ by the J-functional of u_ν in $(L_1(\mathcal{N}, \mathrm{tr}_{\mathcal{N}}), L_2(\mathcal{N}, \mathrm{tr}_{\mathcal{N}}))$, which is bounded by $C(p)\|x\|_{H_p^c}$ by (3.5.9). Then applying the Stein projection we get three elements a, b, c in $L_2(\mathcal{N})$ such that

$$y = a + b + c$$

and

$$\|a\|_{h_p^d(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} + \|b\|_{h_p^c(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} + \|c\|_{h_p^r(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} \leq C(p)\|x\|_{H_p^c},$$

$$\max\{\|a\|_{L_2(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})}, \|b\|_{L_2(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})}, \|c\|_{L_2(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})}\} \leq f(p, \|x\|_{H_p^c}, \|x\|_2).$$

Now we deduce a decomposition of x satisfying (ii) and (iii) as follows. We consider the following projections in $\mathcal{N}$

$$e = \sum_{n \geq 0} e_{n,n} \otimes 1 \quad \text{and} \quad f = e_{0,0} \otimes 1.$$

Since y has a column structure we have $y = eyf$, hence $y = eaf + ebf + ecf$. Writing

$$a = \sum_{k,n \geq 0} e_{k,n} \otimes a_{k,n} \quad , \quad b = \sum_{k,n \geq 0} e_{k,n} \otimes b_{k,n} \quad \text{and} \quad c = \sum_{k,n \geq 0} e_{k,n} \otimes c_{k,n},$$

we have

$$eaf = \sum_{n \geq 0} e_{n,0} \otimes a_{n,0} \quad , \quad ebf = \sum_{n \geq 0} e_{n,0} \otimes b_{n,0} \quad \text{and} \quad ecf = \sum_{n \geq 0} e_{n,0} \otimes c_{n,0}.$$

Since $d_n(y) = e_{n,0} \otimes d_n(x)$ we get

$$d_n(x) = d_n(a_{n,0}) + d_n(b_{n,0}) + d_n(c_{n,0}) \quad \forall\, n \geq 0.$$

Finally we set

$$\alpha = \sum_{n \geq 0} d_n(a_{n,0}) \quad , \quad \beta = \sum_{n \geq 0} d_n(b_{n,0}) \quad , \quad \gamma = \sum_{n \geq 0} d_n(c_{n,0}),$$

and obtain three elements in $L_2(\mathcal{M})$ such that $x = \alpha + \beta + \gamma$. It is clear that α, β and γ verify the L_2-estimate (iii). Note that here we want a decomposition of x in two elements. We will show that $\alpha \in h_p^d$, $\beta \in h_p^c$ and that the third element γ is in the diagonal space h_p^d. Let us first observe that since $e, f \in \mathcal{N}_0 = \mathcal{B}(\ell_2^{m+1}) \overline{\otimes} \mathcal{M}_0$, we deduce from the module property that

$$\|eaf\|_{h_p^d(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} + \|ebf\|_{h_p^c(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} + \|ecf\|_{h_p^r(\mathcal{N}, \mathrm{tr}_{\mathcal{N}})} \leq C(p)\|x\|_{H_p^c}. \tag{3.5.13}$$

Indeed, the estimate of the first term comes from the fact that e and f are projections, and for the second term we write

$$\begin{aligned} \mathcal{E}_{n-1}|d_n(ebf)|^2 &= \mathcal{E}_{n-1}|ed_n(b)f|^2 = \mathcal{E}_{n-1}(fd_n(b)^* ed_n(b)f) \\ &= f\mathcal{E}_{n-1}(d_n(b)^* ed_n(b))f \leq f\mathcal{E}_{n-1}|d_n(b)|^2 f. \end{aligned}$$

Then $\|ebf\|_{h_p^c(\mathcal{N},\mathrm{tr}_\mathcal{N})} \leq \|b\|_{h_p^c(\mathcal{N},\mathrm{tr}_\mathcal{N})}$. The third term is similar. For the term α we have

$$\begin{aligned}
\|\alpha\|_{h_p^d} &= \Big(\sum_n \|d_n(a_{n,0})\|_p^p\Big)^{1/p} = \Big(\sum_n \|(|d_n(a_{n,0})|^2)^{1/2}\|_p^p\Big)^{1/p} \\
&\leq \Big(\sum_n \Big\|\Big(\sum_k |d_n(a_{k,0})|^2\Big)^{1/2}\Big\|_p^p\Big)^{1/p} = \Big(\sum_n \Big\|\sum_k e_{k,0}\otimes d_n(a_{k,0})\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})}^p\Big)^{1/p} \\
&= \Big(\sum_n \|d_n(eaf)\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})}^p\Big)^{1/p} = \|eaf\|_{h_p^d(\mathcal{N},\mathrm{tr}_\mathcal{N})}.
\end{aligned}$$

We proceed similarly for the term β

$$\begin{aligned}
\|\beta\|_{h_p^c} &= \Big\|\Big(\sum_n \mathcal{E}_{n-1}|d_n(b_{n,0})|^2\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sum_{n,k} \mathcal{E}_{n-1}|d_n(b_{k,0})|^2\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\sum_n \mathcal{E}_{n-1}\Big|\sum_k e_{k,0}\otimes d_n(b_{k,0})\Big|^2\Big)^{1/2}\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})} \\
&= \Big\|\Big(\sum_n \mathcal{E}_{n-1}|d_n(ebf)|^2\Big)^{1/2}\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})} = \|ebf\|_{h_p^c(\mathcal{N},\mathrm{tr}_\mathcal{N})}.
\end{aligned}$$

Finally for the term γ we write

$$\begin{aligned}
\|\gamma\|_{h_p^d} &= \Big(\sum_n \|d_n(c_{n,0})\|_p^p\Big)^{1/p} = \Big\|\sum_n e_{n,n}\otimes d_n(c_{n,0})\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})} \\
&= \Big\|\mathrm{Diag}\Big(\sum_{k,n} e_{k,n}\otimes d_n(c_{k,0})\Big)\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})},
\end{aligned}$$

where Diag denotes the diagonal projection in $\mathcal{N}$. Since the diagonal projection is bounded on $L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})$, it remains to estimate

$$\begin{aligned}
\Big\|\sum_{k,n} e_{k,n}\otimes d_n(c_{k,0})\Big\|_{L_p(\mathcal{N},\mathrm{tr}_\mathcal{N})} &= \Big\|\sum_{k,n} e_{0,n}\otimes e_{k,0}\otimes d_n(c_{k,0})\Big\|_{L_p(\mathcal{B}(\ell_2^{m+1})\overline{\otimes}\mathcal{N},\mathrm{tr}\otimes\mathrm{tr}_\mathcal{N})} \\
&= \Big\|\sum_{k,n} e_{0,n}\otimes d_n(ecf)\Big\|_{L_p(\mathcal{B}(\ell_2^{m+1})\overline{\otimes}\mathcal{N},\mathrm{tr}\otimes\mathrm{tr}_\mathcal{N})} \\
&= \|ecf\|_{h_p^r(\mathcal{N},\mathrm{tr}_\mathcal{N})}.
\end{aligned}$$

Then, using (3.5.13), we deduce (ii) and the Theorem follows for the decomposition

$$x = (\alpha+\gamma)+\beta.$$

□

These results can be translated using the $\boxplus$-sum as follows.

Corollary 3.5.7. *Let $1 < p < 2$. Then*

(i) $L_p(\mathcal{M}) = H_p^c \boxplus H_p^r$,

(ii) $H_p^c = h_p^d \boxplus h_p^c$,

with equivalent norms.

Proof. The proofs are similar. Let us detail the proof of (i). In this application we consider $A_0 = L_2(\mathcal{M}), X = H_p^c, Y = H_p^r$ and $A_1 = L_p(\mathcal{M})$. The density assumption (3.5.1) is satisfied, and we are under the condition of Remark 3.5.2. By the density of $L_2(\mathcal{M})$, it suffices to see that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\!|\cdot|\!\|_{H_p^c \boxplus H_p^r}$ defined for $x \in L_2(\mathcal{M})$ by

$$\|\!|x|\!\|_{H_p^c \boxplus H_p^r} = \inf_{x=a+b, a,b \in L_2(\mathcal{M})} \|a\|_{H_p^c} + \|b\|_{H_p^r}.$$

Theorem 3.5.5 (1) immediately implies for $x \in L_2(\mathcal{M})$

$$\|x\|_p \leq \|\!|x|\!\|_{H_p^c \boxplus H_p^r} \leq C(p)\|x\|_p,$$

which ends the proof of (i). For the proof of (ii) we use the same argument and Theorem 3.5.5 (2). □

3.5.3 The $\boxplus$-Burkholder-Gundy inequalities for $1 < p < 2$

The aim of this subsection is to establish the analogue of Corollary 3.5.7 (i) in the setting of a continuous filtration. In this application we consider

$$A_0 = L_2(\mathcal{M}),\ X = \mathcal{H}_p^c,\ Y = \mathcal{H}_p^r \ \text{ and } \ A_1 = L_p(\mathcal{M}).$$

The definitions of $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$ ensure that the density assumption (3.5.1) is satisfied, and by Proposition 3.2.33 the injectivity conditions hold true, i.e., $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$ embed into $L_p(\mathcal{M})$. A direct application of Theorem 3.5.5 gives the following result.

Theorem 3.5.8. *Let $1 < p < 2$. Then*

$$L_p(\mathcal{M}) = \mathcal{H}_p^c \boxplus \mathcal{H}_p^r \quad \textit{with equivalent norms.}$$

Proof. By the density of $L_2(\mathcal{M})$, it suffices to show that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\!|\cdot|\!\|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r}$ defined for $x \in L_2(\mathcal{M})$ by

$$\|\!|x|\!\|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r} = \inf_{x=a+b, a,b \in L_2(\mathcal{M})} \|a\|_{\mathcal{H}_p^c} + \|b\|_{\mathcal{H}_p^r}.$$

Let $x \in L_2(\mathcal{M})$. Then (3.2.2) immediately implies the inequality

$$\|x\|_p \leq \beta_p \|\!|x|\!\|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r}.$$

Conversely, let us suppose $\|x\|_p \leq 1$. Applying Theorem 3.5.5 to each partition σ, we obtain elements $a(\sigma), b(\sigma) \in L_2(\mathcal{M})$ verifying

(i) $x = a(\sigma) + b(\sigma)$,

(ii) $\|a(\sigma)\|_{H_p^c(\sigma)} + \|b(\sigma)\|_{H_p^r(\sigma)} \leq C(p)$,

(iii) $\max\{\|a(\sigma)\|_2, \|b(\sigma)\|_2\} \leq f(p, \|x\|_p, \|x\|_2)$.

Note that the bound of the L_2-norms does not depend on the partition σ. Hence the families $(a(\sigma))_\sigma$ and $(b(\sigma))_\sigma$ are uniformly bounded in $L_2(\mathcal{M})$, and we can consider the weak-limits in L_2

$$a = w\text{-}\lim_{\sigma,\mathcal{U}} a(\sigma) \quad \text{and} \quad b = w\text{-}\lim_{\sigma,\mathcal{U}} b(\sigma).$$

Then we may write

$$x = \mathcal{E}_{\mathcal{U}}(i_{\mathcal{U}}(x)) = \mathcal{E}_{\mathcal{U}}\big((a(\sigma))^{\bullet}\big) + \mathcal{E}_{\mathcal{U}}\big((b(\sigma))^{\bullet}\big) = a + b,$$

where $a, b \in L_2(\mathcal{M})$ satisfy by Proposition 3.2.38

$$\begin{aligned}\|a\|_{\mathcal{H}_p^c} + \|b\|_{\mathcal{H}_p^r} &\leq \beta_p\big(\|(a(\sigma))^{\bullet}\|_{\mathcal{H}_p^c(\mathcal{U})} + \|(b(\sigma))^{\bullet}\|_{\mathcal{H}_p^r(\mathcal{U})}\big)\\ &= \beta_p \lim_{\sigma,\mathcal{U}} \big(\|a(\sigma)\|_{H_p^c(\sigma)} + \|b(\sigma)\|_{H_p^r(\sigma)}\big)\\ &\leq \beta_p C(p).\end{aligned}$$

Thus

$$\|x\|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r} \leq \beta_p C(p) \|x\|_p.$$

□

Theorem 3.2.56 implies

Corollary 3.5.9. *Let* $1 < p < 2$. *Then*

$$\mathcal{H}_p = \mathcal{H}_p^c \boxplus \mathcal{H}_p^r \quad \textit{isometrically.}$$

Proof. Combining Theorem 3.5.8 with Theorem 3.2.56 we immediately obtain that

$$\mathcal{H}_p = \mathcal{H}_p^c + \mathcal{H}_p^r = \mathcal{H}_p^c \boxplus \mathcal{H}_p^r.$$

But the constant depends on $C(p)$. However, this equality implies that the quotient map $q : \mathcal{H}_p^c \boxplus \mathcal{H}_p^r \to \mathcal{H}_p^c + \mathcal{H}_p^r$ is injective. Then we obtain that the two sums coincide isometrically. □

By approximation, we deduce the same result for $p = 1$.

Theorem 3.5.10. *We have*

$$\mathcal{H}_1 = \mathcal{H}_1^c \boxplus \mathcal{H}_1^r \quad \textit{isometrically}\ .$$

Proof. By the density of $L_2(\mathcal{M})$, it suffices to show that the norm $\|\cdot\|_{\mathcal{H}_1}$ is equivalent to the norm $|\!|\!| \cdot |\!|\!|_{\mathcal{H}_1^c \boxplus \mathcal{H}_1^r}$ defined for $x \in L_2(\mathcal{M})$ by

$$|\!|\!| x |\!|\!|_{\mathcal{H}_1^c \boxplus \mathcal{H}_1^r} = \inf_{x=a+b, a,b \in L_2(\mathcal{M})} \|a\|_{\mathcal{H}_1^c} + \|b\|_{\mathcal{H}_1^r}.$$

Let $x \in L_2(\mathcal{M})$. We clearly have the inequality

$$\|x\|_{\mathcal{H}_1} = \|q(x)\|_{\mathcal{H}_1^c + \mathcal{H}_1^r} \leq |\!|\!| x |\!|\!|_{\mathcal{H}_1^c \boxplus \mathcal{H}_1^r}.$$

Conversely, let us suppose $\|x\|_{\mathcal{H}_1} < 1$. Then by Theorem 3.2.62 we have

$$\|x\|_{\check{\mathcal{H}}_1} = \lim_{p \to 1} \|x\|_{\mathcal{H}_p} < 1.$$

Hence there exits $1 < p < 2$ such that $\|x\|_{\mathcal{H}_p} < 1$. By Corollary 3.5.9 this means that $|\!|\!| x |\!|\!|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r} < 1$. Thus we get

$$|\!|\!| x |\!|\!|_{\mathcal{H}_1^c \boxplus \mathcal{H}_1^r} \leq |\!|\!| x |\!|\!|_{\mathcal{H}_p^c \boxplus \mathcal{H}_p^r} < 1.$$

This ends the proof. □

Hence Lemma 3.5.3 yields the following density result.

Corollary 3.5.11. *Let* $1 \leq p < 2$. *Then* $L_2(\mathcal{M})$ *is dense in the intersection* $\mathcal{H}_p^c \cap \mathcal{H}_p^r$.

3.5.4 The $\boxplus$-Davis decomposition for $1 < p < 2$

This subsection deals with the continuous analogue of Corollary 3.5.7 (ii). Let us consider $1 < p < 2$ and

$$A_0 = L_2(\mathcal{M}),\ X = \mathsf{h}_p^d,\ Y = \mathsf{h}_p^c \ \text{ and } \ A_1 = L_p(\mathcal{M}).$$

The definitions of h_p^d and h_p^c ensure that the density assumption (3.5.1) is satisfied, and by Remark 3.4.13 (2) and Theorem 3.3.29, the injectivity conditions hold true, i.e., h_p^d and h_p^c embed into $L_p(\mathcal{M})$.

Theorem 3.5.12. *Let $1 < p < 2$. Then*

$$\mathcal{H}_p^c = \mathsf{h}_p^d \boxplus \mathsf{h}_p^c = \mathsf{h}_p^d + \mathsf{h}_p^c \qquad \textit{with equivalent norms.}$$

Moreover, the constant remains bounded as $p \to 1$.

Proof. We can adapt the argument detailed in the proof of Theorem 3.5.8 by using the decomposition given by Corollary 3.5.6. This shows that $\mathcal{H}_p^c = \mathsf{h}_p^d \boxplus \mathsf{h}_p^c$, but the constant $C(p)$ does not remain bounded as $p \to 1$. However, by Theorem 3.4.24 we know that $\mathcal{H}_p^c = \mathsf{h}_p^d + \mathsf{h}_p^c$, and since this still holds true for $p = 1$, the constant remains bounded as $p \to 1$. Thus the two sums coincide, and the quotient map $q : \mathsf{h}_p^d \boxplus \mathsf{h}_p^c \to \mathsf{h}_p^d + \mathsf{h}_p^c$ is injective. We obtain that $\mathsf{h}_p^d \boxplus \mathsf{h}_p^c = \mathsf{h}_p^d + \mathsf{h}_p^c$ isometrically, and the constant in the equivalence $\mathcal{H}_p^c = \mathsf{h}_p^d \boxplus \mathsf{h}_p^c$ is the same than the constant in Theorem 3.4.24, hence remains bounded as $p \to 1$. $\square$

3.5.5 The Burkholder inequalities for $2 < p < \infty$

Let us first introduce the following seminorm. For $2 < p \leq \infty$ and $x \in L_p(\mathcal{M})$ we define

$$\|x\|_{\mathsf{h}_p^d} = \lim_{\sigma,\mathcal{U}} \|x\|_{h_p^d(\sigma)}.$$

By interpolation between the cases $p = 2$ and $p = \infty$, we have

$$\|x\|_{\mathsf{h}_p^d} \leq 2\|x\|_p.$$

In this case we also have some monotonicity properties.

Lemma 3.5.13. *Let $2 < p \leq \infty$, $x \in L_p(\mathcal{M})$ and $\sigma \subset \sigma'$. Then*

$$\|x\|_{h_p^d(\sigma')} \leq 2\|x\|_{h_p^d(\sigma)}.$$

Hence

$$\frac{1}{2}\|x\|_{\mathsf{h}_p^d} \leq \inf_{\sigma} \|x\|_{h_p^d(\sigma)} \leq \|x\|_{\mathsf{h}_p^d}.$$

Proof. The proof is similar to that of Lemma 3.4.9, hence we omit the details. $\square$

As a direct consequence, we see that $\|\cdot\|_{\mathsf{h}_p^d}$ does not depend, up to a constant, on the choice of the ultrafilter $\mathcal{U}$. The goal of this subsection is to state the missing Burkholder inequalities for $2 < p < \infty$. We will first establish an analogue of Theorem 3.4.1 for $2 < p < \infty$ by a dual approach. More precisely, we would like to consider the dual version of Theorem 3.5.12. The delicate point here is to describe the dual space of the diagonal space h_p^d for $1 < p < 2$. Since we are only interested in the dual of the sum $\mathsf{h}_p^d \boxplus \mathsf{h}_p^c$, the key trick is to replace h_p^d in this sum by a nicer space, without changing the $\boxplus$-sum. We first observe that since $L_2(\mathcal{M})$ is dense in h_p^c, we have

Lemma 3.5.14. *Let* $1 \leq p < 2$. *Then*

$$\mathsf{h}_p^c = L_2(\mathcal{M}) \boxplus \mathsf{h}_p^c \quad \textit{isometrically.}$$

Proof. We consider

$$A_0 = L_2(\mathcal{M}),\ X = L_2(\mathcal{M}),\ Y = \mathsf{h}_p^c \text{ and } A_1 = L_p(\mathcal{M}).$$

By the density of $L_2(\mathcal{M})$ in h_p^c it suffices to see that $\|x\|_{\mathsf{h}_p^c} = \|x\|_{L_2(\mathcal{M})\boxplus\mathsf{h}_p^c}$ for all $x \in L_2(\mathcal{M})$. Let $x \in L_2(\mathcal{M})$. It is clear that $\|x\|_{L_2(\mathcal{M})\boxplus\mathsf{h}_p^c} \leq \|x\|_{\mathsf{h}_p^c}$. Conversely, we assume $\|x\|_{L_2(\mathcal{M})\boxplus\mathsf{h}_p^c} < 1$. Then there exist $a, b \in L_2(\mathcal{M})$ such that

$$x = a + b \quad \text{and} \quad \|a\|_2 + \|b\|_{\mathsf{h}_p^c} < 1.$$

By the Hölder inequality we get

$$\|x\|_{\mathsf{h}_p^c} \leq \|a\|_{\mathsf{h}_p^c} + \|b\|_{\mathsf{h}_p^c} \leq \|a\|_2 + \|b\|_{\mathsf{h}_p^c} < 1.$$

□

The idea is to add the space $L_2(\mathcal{M})$ to h_p^d to obtain a new larger diagonal space, in which $L_2(\mathcal{M})$ will be dense, and which will preserve the $\boxplus$-sum with h_p^c. Hence we introduce the following space, which will play the role of h_p^d in the sequel.

Definition 3.5.15. *Let* $1 \leq p < 2$. *We define*

$$\mathrm{K}_p^d = \mathsf{h}_p^d \boxplus L_2(\mathcal{M}),$$

i.e., K_p^d *is the completion of* $L_2(\mathcal{M})$ *with respect to the norm*

$$\|x\|_{\mathrm{K}_p^d} = \inf_{x=a+b, a\in L_2(\mathcal{M})\cap\mathsf{h}_p^d, b\in L_2(\mathcal{M})} \|a\|_{\mathsf{h}_p^d} + \|b\|_2.$$

Note that in this application we consider

$$A_0 = L_2(\mathcal{M}),\ X = \mathsf{h}_p^d,\ Y = L_2(\mathcal{M}) \text{ and } A_1 = L_p(\mathcal{M}).$$

By the definition of h_p^d, these spaces satisfy (3.5.1) and the injectivity assumtion is also verified. We need to work a little bit to show that this space embeds into $L_p(\mathcal{M})$. The discrete analogue of K_p^d is the space $K_p^d(\sigma) = h_p^d(\sigma) \boxplus L_2(\mathcal{M})$, defined as the completion of $L_2(\mathcal{M})$ with respect to the norm

$$\|x\|_{K_p^d(\sigma)} = \inf_{x=a+b, a\in L_2(\mathcal{M}), b\in L_2(\mathcal{M})} \|a\|_{h_p^d(\sigma)} + \|b\|_2.$$

Observe that since we consider finite partitions, the norm $\|\cdot\|_{h_p^d(\sigma)}$ is equivalent to the norm $\|\cdot\|_p$ for $1 \leq p < 2$. Hence $K_p^d(\sigma)$ is $L_p(\mathcal{M})$ equipped with the norm $\|\cdot\|_{K_p^d(\sigma)}$.

Lemma 3.5.16. *Let* $1 \leq p < 2$ *and* $x \in L_2(\mathcal{M})$. *Then*

$$\frac{1}{2}\|x\|_{\mathrm{K}_p^d} \leq \lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)} \leq \|x\|_{\mathrm{K}_p^d}.$$

Moreover we have a contractive injective map

$$i_{\mathcal{U}} : K_p^d \to \prod\nolimits_{\mathcal{U}} K_p^d(\sigma).$$

Proof. Let $x \in L_2(\mathcal{M})$. It is obvious that

$$\lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)} \le \|x\|_{\mathrm{K}_p^d}.$$

Conversely, we assume $\lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)} < 1$. We may suppose that $\|x\|_{K_p^d(\sigma)} < 1$ for all σ. Then for each σ there exist $a(\sigma), b(\sigma) \in L_2(\mathcal{M})$ such that

$$x = a(\sigma) + b(\sigma) \quad \text{and} \quad \|a(\sigma)\|_{h_p^d(\sigma)} + \|b(\sigma)\|_2 < 1.$$

Note that

$$\|a(\sigma)\|_2 = \|x - b(\sigma)\|_2 \le \|x\|_2 + 1.$$

Hence the families $(a(\sigma))_\sigma$ and $(b(\sigma))_\sigma$ are uniformly bounded in $L_2(\mathcal{M})$, and we can consider the weak-limits in L_2

$$a = w\text{-}\lim_{\sigma,\mathcal{U}} a(\sigma) \quad \text{and} \quad b = w\text{-}\lim_{\sigma,\mathcal{U}} b(\sigma).$$

Then we may write

$$x = \mathcal{E}_{\mathcal{U}}(i_{\mathcal{U}}(x)) = \mathcal{E}_{\mathcal{U}}\big((a(\sigma))^\bullet\big) + \mathcal{E}_{\mathcal{U}}\big((b(\sigma))^\bullet\big) = a + b,$$

where $a \in L_2(\mathcal{M}) \cap \mathsf{h}_p^d, b \in L_2(\mathcal{M})$ satisfy by Lemma 3.4.19

$$\|a\|_{\mathsf{h}_p^d} + \|b\|_2 \le 2 \lim_{\sigma,\mathcal{U}} \big(\|a(\sigma)\|_{h_p^d(\sigma)} + \|b(\sigma)\|_2\big) \le 2.$$

We obtain

$$\|x\|_{\mathrm{K}_p^d} \le 2 \lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)}.$$

Finally, since the norms are equivalent, it is clear that the map $i_{\mathcal{U}}$, defined on $L_2(\mathcal{M})$ by $i_{\mathcal{U}}(x) = (x)^\bullet$, extends to an injective map on K_p^d. $\square$

Note that by Lemma 3.4.9 we have

$$\|x\|_{K_p^d(\sigma)} \le 2\|x\|_{K_p^d(\sigma')}$$

for $\sigma \subset \sigma'$ and $x \in L_2(\mathcal{M})$. Hence

$$\lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)} \le \sup_{\sigma} \|x\|_{K_p^d(\sigma)} \le 2 \lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)}.$$

Thus adapting the proof of Proposition 3.2.33 and using Lemma 3.5.16 we can show that

Lemma 3.5.17. *Let $1 \le p < 2$. Then*

(i) $\{x \in L_p(\mathcal{M}) \ : \ \|x\|_{\mathrm{K}_p^d} < \infty\}$ *is complete.*

(ii) K_p^d *embeds injectively into* $L_p(\mathcal{M})$.

Observe that by Lemma 3.5.3, we deduce that in fact $\mathrm{K}_p^d = \mathsf{h}_p^d + L_2(\mathcal{M})$ isometrically. We can now consider

$$A_0 = L_2(\mathcal{M}),\ X = \mathrm{K}_p^d,\ Y = \mathsf{h}_p^c \ \text{ and } \ A_1 = L_p(\mathcal{M}).$$

Indeed, Lemma 3.5.17 ensures that X embeds into A_1, and we are in the situation described in subsection 3.5.1. Then, setting $Z = L_2(\mathcal{M})$, the associative relation given in Remark 3.5.2 combined with Lemma 3.5.14 yields that K_p^d preserves the $\boxplus$-sum with h_p^c in the following sense.

Lemma 3.5.18. *Let $1 \leq p < 2$. Then*

$$\mathsf{h}_p^d \boxplus \mathsf{h}_p^c = \mathrm{K}_p^d \boxplus \mathsf{h}_p^c \quad \textit{isometrically.}$$

Let us now consider the dual situation. Let $2 < p \leq \infty$ and σ be a finite partition. Let $J_p^d(\sigma)$ be the space $L_p(\mathcal{M})$ equipped with the norm

$$\|x\|_{J_p^d(\sigma)} = \max(\|x\|_{h_p^d(\sigma)}, \|x\|_2).$$

By Lemma 3.5.13, it is clear that for $2 < p \leq \infty, x \in L_p(\mathcal{M})$ and $\sigma \subset \sigma'$ we have

$$\|x\|_{J_p^d(\sigma')} \leq 2\|x\|_{J_p^d(\sigma)}.$$

For $1 \leq p < 2$, the discrete duality $h_p^d(\sigma)$-$h_{p'}^d(\sigma)$ implies

$$(K_p^d(\sigma))^* = J_{p'}^d(\sigma) \quad \text{with equivalent norms.}$$

Moreover,

$$\frac{1}{2}\|x\|_{J_{p'}^d(\sigma)} \leq \|x\|_{(K_p^d(\sigma))^*} \leq \|x\|_{J_{p'}^d(\sigma)}.$$

In order to describe the dual space of K_p^d, we introduce the following definition, in the same spirit as the definition of the spaces $L_p^c\mathcal{MO}$. Since for each σ we have a contractive inclusion $J_p^d(\sigma) \subset L_2(\mathcal{M})$, by considering the ultraproduct map we get a contractive map from $\prod_{\mathcal{U}} J_p^d(\sigma)$ to $L_2(\widetilde{\mathcal{M}}_{\mathcal{U}})$. Hence taking the weak-limit in L_2 we may define the bounded map $\mathcal{E}_{\mathcal{U}} : \prod_{\mathcal{U}} J_p^d(\sigma) \to L_2(\mathcal{M})$.

Definition 3.5.19. *(i) Let $2 < p < \infty$. We define the space J_p^d as the quotient space of $\prod_{\mathcal{U}} J_p^d(\sigma)$ by the kernel of the map $\mathcal{E}_{\mathcal{U}}$. The norm in J_p^d is given by the usual quotient norm*

$$\|x\|_{\mathrm{J}_p^d} = \inf_{x = w\text{-}\lim_\sigma x_\sigma} \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{J_p^d(\sigma)}.$$

(ii) We define the space J_∞^d as the space whose closed unit ball is given by the absolute convex set

$$B_{\mathrm{J}_\infty^d} = \overline{\{x = w\text{-}\lim_\sigma x_\sigma \text{ in } L_2 \; : \; \lim_{\sigma,\mathcal{U}} \|x_\sigma\|_{J_\infty^d(\sigma)} \leq 1\}}^{\|\cdot\|_2}.$$

Then the norm in J_∞^d is given by

$$\|x\|_{\mathrm{J}_\infty^d} = \inf\{C \geq 0 \; : \; x \in CB_{\mathrm{J}_\infty^d}\}.$$

It is clear that for $2 < p < \infty$, the space J_p^d is complete. For $p = \infty$, J_∞^d is a Banach space by Lemma 3.2.45. We may characterize the space J_p^d in a simpler way.

Proposition 3.5.20. *Let $2 < p \leq \infty$. Then the unit ball of J_p^d is equivalent to*

$$\mathbb{B}_p = \{x \in L_2(\mathcal{M}) \; : \; x = L_2\text{-}\lim_\lambda x_\lambda, \; \|x_\lambda\|_{\mathsf{h}_p^d} \leq 1, \; \|x_\lambda\|_2 \leq 1, \; \forall \, \lambda\}.$$

Proof. Since the discrete $J_p^d(\sigma)$-norms are decreasing in σ (up to a constant 2), we may adapt the proof of Proposition 3.2.52 and obtain that the unit ball of J_p^d is equivalent to

$$\{x \in L_2(\mathcal{M}) \; : \; x = L_2\text{-}\lim_\lambda x_\lambda, \lim_{\sigma,\mathcal{U}} \|x_\lambda\|_{J_p^d(\sigma)} \leq 1, \; \forall \, \lambda\}.$$

Moreover, it is clear that for $x \in L_p(\mathcal{M})$

$$\lim_{\sigma,\mathcal{U}} \|x\|_{J_p^d(\sigma)} \simeq_2 \max(\|x\|_{\mathsf{h}_p^d}, \|x\|_2).$$

Hence we get

$$\frac{1}{2} B_{\mathrm{J}_p^d} \subset \mathbb{B}_p \subset 2B_{\mathrm{J}_p^d}.$$

□

This construction describes the dual space of K_p^d.

Lemma 3.5.21. *Let $1 \leq p < 2$. Then*

$$(\mathrm{K}_p^d)^* = \mathrm{J}_{p'}^d \quad \textit{with equivalent norms.}$$

Moreover,

$$\frac{1}{8}\|x\|_{\mathrm{J}_{p'}^d} \leq \|x\|_{(\mathrm{K}_p^d)^*} \leq \|x\|_{\mathrm{J}_{p'}^d}.$$

Proof. The proof is similar to that of Theorem 3.2.46. Indeed the construction of the space $\mathrm{J}_{p'}^d$ is similar to that of the space $L_{p'}^c\mathcal{MO}$. The contractive inclusion $\mathrm{J}_{p'}^d \subset (\mathrm{K}_p^d)^*$ follows easily from the discrete duality $(K_p^d(\sigma))^* = J_{p'}^d(\sigma)$ and the density of $L_2(\mathcal{M})$ in K_p^d. For the reverse inclusion, recall that by Lemma 3.5.16 the space K_p^d embeds into $\prod_{\mathcal{U}} K_p^d(\sigma)$, and $\|x\|_{\mathrm{K}_p^d} \leq 2\lim_{\sigma,\mathcal{U}} \|x\|_{K_p^d(\sigma)}$. Hence by the Hahn-Banach Theorem we may extend a linear functional on K_p^d of norm less than one to a linear functional on $\prod_{\mathcal{U}} K_p^d(\sigma)$ of norm less than two. Then we use the same argument as in the proof of Theorem 3.2.46. The crucial point here is that

$$L_2(\mathcal{M}) \text{ is dense in } \mathrm{K}_p^d \quad \text{and} \quad \|x\|_2 \leq \|x\|_{J_p^d(\sigma)}. \tag{3.5.14}$$

□

Remark 3.5.22. The same argument doesn't work if the observation (3.5.14) is not verified. This explains why we cannot easily describe similarly the dual space of h_p^d for $1 \leq p < 2$, and justifies the introduction of the spaces K_p^d.

We can now establish the dual version of Theorem 3.5.12.

Theorem 3.5.23. *Let $2 < p < \infty$. Then*

$$\mathcal{H}_p^c = \mathrm{J}_p^d \cap \mathsf{h}_p^c \quad \textit{with equivalent norms.}$$

Moreover, the constant remains bounded as $p \to \infty$.

Proof. Combining Theorem 3.5.12 with Lemma 3.5.18 and Theorem 3.2.39 we get for $2 < p \leq \infty$

$$\mathcal{H}_p^c = (\mathcal{H}_{p'}^c)^* = (\mathsf{h}_{p'}^d \boxplus \mathsf{h}_{p'}^c)^* = (\mathrm{K}_{p'}^d \boxplus \mathsf{h}_{p'}^c)^*.$$

Then, using the description of the dual space of the sum $\boxplus$ given in Lemma 3.5.4, Lemma 3.5.21 and Theorem 3.3.29 yield

$$\mathcal{H}_p^c = (\mathrm{K}_{p'}^d)^* \Cap (\mathsf{h}_{p'}^c)^* = \mathrm{J}_p^d \Cap \mathsf{h}_p^c.$$

Note that here the intersection is taken over $L_2(\mathcal{M}) \cap \mathrm{K}_{p'}^d \cap \mathsf{h}_{p'}^c = L_2(\mathcal{M})$. Moreover, J_p^d and h_p^c are subspaces of $L_2(\mathcal{M})$. Hence in this situation, $\Cap$ is simply the usual intersection of subspaces in $L_2(\mathcal{M})$, and we can denote it by $\cap$. This concludes the proof. □

We turn to the missing Burkholder inequalities for $2 < p < \infty$. We introduce the conditioned Hardy space h_p in this case as follows.

Definition 3.5.24. *Let $2 < p < \infty$. We define*

$$\mathsf{h}_p = \mathrm{J}_p^d \cap \mathsf{h}_p^c \cap \mathsf{h}_p^r,$$

equipped with the intersection norm

$$\|x\|_{\mathsf{h}_p} = \max\left(\|x\|_{\mathrm{J}_p^d}\ ,\ \|x\|_{\mathsf{h}_p^c}\ ,\ \|x\|_{\mathsf{h}_p^r}\right).$$

Theorem 3.5.25. *Let $2 < p < \infty$. Then*

$$L_p(\mathcal{M}) = \mathsf{h}_p \quad \textit{with equivalent norms.}$$

Proof. Combining Theorem 3.2.56 with Theorem 3.5.23 we get for $2 < p < \infty$

$$L_p(\mathcal{M}) = \mathcal{H}_p^c \cap \mathcal{H}_p^r = (\mathrm{J}_p^d \cap \mathsf{h}_p^c) \cap (\mathrm{J}_p^d \cap \mathsf{h}_p^r) = \mathsf{h}_p.$$

□

3.5.6 Another characterization of $\mathcal{BMO}^c$ and the ⊞-Davis decomposition for $p = 1$

We end this section with the case $p = 1$. The aim is to extend Theorem 3.5.12 to the case $p = 1$. Observe that in this case, since the diagonal spaces $h_p^d(\sigma)$ are not regular, we can not prove the case $p = 1$ by approximation, as we did in Theorem 3.5.10. Hence we will use a dual approach and will first extend Theorem 3.5.23 to the case $p' = \infty$. More precisely, by approximation we can establish an analogue of the discrete fact that $BMO^c(\sigma) = h_\infty^d(\sigma) \cap bmo^c(\sigma)$ as follows.

Theorem 3.5.26. *We have*

$$(\mathcal{H}_1^c)^* = \mathcal{BMO}^c = \mathrm{J}_\infty^d \cap \mathsf{bmo}^c \quad \textit{with equivalent norms.}$$

Proof. We will prove the following inclusions

$$\mathcal{BMO}^c \overset{(1)}{\subset} \mathrm{J}_\infty^d \cap \mathsf{bmo}^c \overset{(2)}{\subset} (\mathcal{H}_1^c)^*.$$

Then we will conclude by using Theorem 3.2.46. For each σ and $x \in \mathcal{M}$ we have $\|x\|_{h_\infty^d(\sigma)} \leq \|x\|_{BMO^c(\sigma)}$ and $\|x\|_2 \leq \sqrt{2}\|x\|_{BMO^c(\sigma)}$. Hence $B_{\mathcal{BMO}^c} \subset \sqrt{2}B_{\mathrm{J}_\infty^d}$. Moreover, $\|x\|_{bmo^c(\sigma)} \leq \|x\|_{BMO^c(\sigma)}$. Then Lemma 3.3.36 yields a bounded inclusion $\mathcal{BMO}^c \subset \mathsf{bmo}^c$. This shows (1). We prove (2) by approximation. Let $x \in \mathrm{J}_\infty^d \cap \mathsf{bmo}^c$ be of norm less than one and $y \in L_2(\mathcal{M})$. Then $x \in L_2(\mathcal{M})$ satisfies $\max(\|x\|_{\mathrm{J}_\infty^d}, \|x\|_{\mathsf{bmo}^c}) \leq 1$. Hence x is in the unit ball of J_∞^d, and by Lemma 3.5.20 there exists a sequence $(x_\lambda)_\lambda$ in $\mathcal{M}$ such that $x = L_2\text{-}\lim_\lambda x_\lambda$ and $\|x_\lambda\|_{\mathsf{h}_\infty^d} \leq 2, \|x_\lambda\|_2 \leq 2$. On the other hand, by Lemma 3.2.25 we have

$$\|y\|_{\mathcal{H}_1^c} = \lim_{p\to 1} \|y\|_{\mathcal{H}_p^c}.$$

Hence for $\varepsilon > 0$, there exists $1 < p < 2$ such that $\|y\|_{\mathcal{H}_p^c} \leq \|y\|_{\mathcal{H}_1^c} + \varepsilon$. Observe that for $2 < p' < \infty$ the conjugate index of p we have for each σ

$$\|x_\lambda\|_{h_{p'}^d(\sigma)} \leq 2\|x_\lambda\|_{h_\infty^d(\sigma)}^{1-2/p'} \|x_\lambda\|_{h_2^d(\sigma)}^{2/p'} \leq 2^{1+2/p'} \|x_\lambda\|_{h_\infty^d(\sigma)}^{1-2/p'}.$$

Taking the limit in σ we get

$$\|x_\lambda\|_{\mathsf{h}^d_{p'}} \leq 2^{1+2/p'} \|x_\lambda\|_{\mathsf{h}^d_\infty}^{1-2/p'} \leq 4.$$

Hence $x \in \mathrm{J}^d_{p'}$. Since $\mathsf{bmo}^c \subset L^c_{p'}\mathsf{mo}$, we get

$$x \in \mathrm{J}^d_{p'} \cap L^c_{p'}\mathsf{mo} = (\mathrm{K}^d_p \boxplus \mathsf{h}^c_p)^* = (\mathsf{h}^d_p \boxplus \mathsf{h}^c_p)^* = (\mathcal{H}^c_p)^*.$$

It is crucial here to note that the constant remains bounded as $p \to 1$, thanks to Theorem 3.5.12. Thus

$$|\tau(x^*y)| \leq C(p)\|y\|_{\mathcal{H}^c_p} \leq C(\|y\|_{\mathcal{H}^c_1} + \varepsilon),$$

where $C = \sup_{1<p<4/3} C(p)$ is bounded. Sending ε to 0 we obtain that $x \in (\mathcal{H}^c_1)^*$, and (2) is proved. □

This duality allows us to decompose the space $\mathcal{H}^c_1$ as follows.

Corollary 3.5.27. *We have*

$$\mathcal{H}^c_1 = \mathrm{K}^d_1 \boxplus \mathsf{h}^c_1 \quad \textit{with equivalent norms.}$$

Proof. This follows directly by duality. Indeed, by the same arguments as in the proof of Theorem 3.5.23, we have

$$(\mathrm{K}^d_1 \boxplus \mathsf{h}^c_1)^* = (\mathrm{K}^d_1)^* \Cap \mathsf{bmo}^c = \mathrm{J}^d_\infty \Cap \mathsf{bmo}^c.$$

Since in this case the intersection $\Cap$ corresponds to the usual intersection of subspaces of $L_2(\mathcal{M})$, we get by Theorem 3.5.26

$$(\mathrm{K}^d_1 \boxplus \mathsf{h}^c_1)^* = \mathrm{J}^d_\infty \cap \mathsf{bmo}^c = \mathcal{BMO}^c.$$

We conclude the proof by using Theorem 3.2.46. □

Combining Corollary 3.5.27 with Lemma 3.5.18, we obtain the $\boxplus$-Davis decomposition for $p = 1$.

Theorem 3.5.28. *We have*

$$\mathcal{H}^c_1 = \mathsf{h}^d_1 \boxplus \mathsf{h}^c_1 \quad \textit{with equivalent norms.}$$

3.6 Atomic decomposition

In this section we introduce the notion of algebraic atoms, and use it to decompose the Hardy spaces defined previously. Actually our sets of algebraic atoms will be already absolutely convex, and we will characterize the unit balls of $\mathcal{H}^c_p$ and h^c_p as the closure (in $\mathcal{H}^c_p$ and h^c_p respectively) of such algebraic atoms. We will use this decomposition in the next section on interpolation.

3.6.1 The discrete case

Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration. In this part we establish a decomposition of the Hardy spaces for $1 \leq p < 2$ involving "algebraic atoms". This decomposition yields an explicit Davis decomposition at the level of algebraic atoms for $1 \leq p < 2$. In Chapter 2, an atomic decomposition for h_1^c was established by a dual approach, and we will follow this procedure to give an algebraic atomic description of H_p^c and h_p^c in the range $1 \leq p < 2$. More precisely, we will first decompose in algebraic atoms the conditioned columns L_p-spaces in which the Hardy spaces are complemented.

Let us consider the conditioned column spaces $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ introduced in [20]. Recall that for $1 \leq p < \infty$ and any finite sequence $a = (a_n)_{n\geq 0}$ in $L_\infty(\mathcal{M})$, we set

$$\|a\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} = \Big\|\Big(\sum_{n\geq 0}\mathcal{E}_n|a_n|^2\Big)^{1/2}\Big\|_p.$$

Then $\|\cdot\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)}$ defines a norm on the family of finite sequences of $L_\infty(\mathcal{M})$. We denote by $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ the corresponding completion. Note that this space is isometrically isomorphic to the space of double indexed columns which are conditioned in one variable considered in [20], by using the map

$$v : \left\{\begin{array}{ccc} L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) & \longrightarrow & L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2)) \\ \sum\limits_{n\geq 0} e_{n,0}\otimes a_n & \longmapsto & \sum\limits_{n\geq 0} e_{n,0}\otimes u_n(a_n) \end{array}\right.$$

with complemented range in

$$L_p^{\mathrm{ad}}(\mathcal{M};\ell_2^c(\mathbb{N}^2)) = \Big\{\sum_{n,k} e_{n,0}\otimes e_{k,0}\otimes a_{n,k} \ :\ a_{n,k}\in L_p(\mathcal{M}_n) \text{ for all } n,k\geq 0\Big\}.$$

It is clear that for $1 \leq p < \infty$, the Hardy space H_p^c embeds isometrically into $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ via the map

$$i : \left\{\begin{array}{ccc} H_p^c & \longrightarrow & L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) \\ x & \longmapsto & \sum\limits_{n\geq 0} e_{n,0}\otimes d_n(x) \end{array}\right. .$$

Moreover, using the Stein projection $\mathcal{D}$, we can show that

Lemma 3.6.1. *Let $1 < p < \infty$. Then H_p^c is complemented in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$.*

Proof. The Stein inequality directly implies that $\|\mathcal{D} : L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) \to H_p^c\| \leq \sqrt{2}(1+\gamma_p)$. Indeed, for $1 < p < \infty$ and a finite sequence $a = (a_n)_n$ in $L_\infty(\mathcal{M})$ we have

$$\begin{aligned}
\|\mathcal{D}(a)\|_{H_p^c} &= \Big\|\Big(\sum_n |\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)|^2\Big)^{1/2}\Big\|_p \\
&\leq \sqrt{2}\Big(\Big\|\Big(\sum_n |\mathcal{E}_n(a_n)|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n |\mathcal{E}_{n-1}(a_n)|^2\Big)^{1/2}\Big\|_p\Big) \\
&= \sqrt{2}\Big(\Big\|\Big(\sum_n |\mathcal{E}_n(a_n)|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_n |\mathcal{E}_{n-1}(\mathcal{E}_n(a_n))|^2\Big)^{1/2}\Big\|_p\Big) \\
&\leq \sqrt{2}(1+\gamma_p)\Big\|\Big(\sum_n |\mathcal{E}_n(a_n)|^2\Big)^{1/2}\Big\|_p \\
&\leq \sqrt{2}(1+\gamma_p)\|a\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)}.
\end{aligned}$$

$\square$

Recall that $\gamma_p \approx (p-1)^{-1}$ as $p \to 1$, hence this direct proof does not hold for $p=1$. However, by using the spirit of the decomposition introduced in section 3.4, we will show that in fact $\mathcal{D}$ is also bounded on $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ for $p=1$.

Let us introduce $L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)$ defined similarly as $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ by setting

$$\|a\|_{L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)} = \Big\| \Big(\sum_{n\geq 0} \mathcal{E}_{n-1}|a_n|^2 \Big)^{1/2} \Big\|_p .$$

We now turn to the description of the dual spaces of these two conditioned L_p-spaces. Following [21] we introduce for $2 < p \leq \infty$

$$L_p^{c,\mathrm{cond}} MO = \{x = (x_n)_{n\geq 0} \subset L_p(\mathcal{M}) \ : \ \|x\|_{L_p^{c,\mathrm{cond}} MO} < \infty\},$$

where

$$\|x\|_{L_p^{c,\mathrm{cond}} MO} = \Big\| {\sup_{n\geq 0}}^{+} \mathcal{E}_n \Big(\sum_{k\geq n} |x_k|^2 \Big) \Big\|_{p/2}^{1/2}.$$

Similarly, we define

$$L_p^{c,\mathrm{cond}-} MO = \{x = (x_n)_{n\geq 0} \subset L_p(\mathcal{M}) \ : \ \|x\|_{L_p^{c,\mathrm{cond}-} MO} < \infty\},$$

where

$$\|x\|_{L_p^{c,\mathrm{cond}-} MO} = \Big\| {\sup_{n\geq 0}}^{+} \mathcal{E}_n \Big(\sum_{k> n} |x_k|^2 \Big) \Big\|_{p/2}^{1/2}.$$

We also define the space $L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)$ as follows. Let $1 \leq p < 2$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. A sequence $x = (x_n)_{n\geq 0}$ is in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)$ if there are $b_{k,n} \in L_2(\mathcal{M})$ and $a_{k,n} \in L_q(\mathcal{M}_n)$ such that

$$x_n = \sum_{k\geq 0} b_{k,n}^* a_{k,n} \tag{3.6.1}$$

for all n and

$$\sum_{k,n\geq 0} |b_{k,n}|^2 \in L_1(\mathcal{M}), \qquad \sum_{k,n\geq 0} |a_{k,n}|^2 \in L_{q/2}(\mathcal{M}).$$

We equip $L_p(\mathcal{M};\ell_1^c)$ with the norm

$$\|x\|_{L_p(\mathcal{M};\ell_1^c)} = \inf \Big\{ \Big(\sum_{k,n\geq 0} \|b_{k,n}\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_{k,n\geq 0} |a_{k,n}|^2 \Big)^{1/2} \Big\|_q \Big\},$$

where the infimum is taken over all factorizations (3.6.1). Observe that we may adapt Lemma 3.4.2 to this conditioned space, i.e., the unit ball of $L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)$ is the set of all sequences $(b_n a_n)_{n\geq 0}$ with $b_n \in L_2(\mathcal{M})$ and $a_n \in L_q(\mathcal{M}_n)$ such that

$$\Big(\sum_{n\geq 0} \|b_n\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_q \leq 1. \tag{3.6.2}$$

For $2 < p \leq \infty$ we consider

$$L_p^{\mathrm{cond}}(\mathcal{M};\ell_\infty^c) = \{x = (x_n)_{n\geq 0} \subset L_p(\mathcal{M}) \ : \ \|x\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_\infty^c)} < \infty\},$$

where

$$\|x\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_\infty^c)} = \Big\| {\sup_{n\geq 0}}^{+} \mathcal{E}_n |x_n|^2 \Big\|_{p/2}^{1/2}.$$

Proposition 3.6.2. *Let $1 \leq p < 2$. Then*

(i) $(L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c))^* = L_{p'}^{c,\mathrm{cond}} MO$,

(ii) $(L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c))^* = L_{p'}^{c,\mathrm{cond}-} MO$,

(iii) $(L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c))^* = L_{p'}^{\mathrm{cond}}(\mathcal{M}; \ell_\infty^c)$,

with equivalent norms.

Proof. (i) and (ii) was proved in [21], by considering $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ as a subspace of $L_p(\mathcal{M}; \ell_2^c(\mathbb{N}^2))$. The proof of (iii) is similar to that of Proposition 3.6 of [20], and is based on a standard application of the Grothendieck-Pietsch version of the Hahn-Banach Theorem. □

It is clear that for a sequence $(x_n)_{n\geq 0}$ in $L_p(\mathcal{M})$ and $2 < p \leq \infty$ we have

$$\|x\|_{L_p^{c,\mathrm{cond}} MO} \simeq \max\big(\|x\|_{L_p^{c,\mathrm{cond}-} MO}\,,\ \|x\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_\infty^c)}\big).$$

Hence by duality we get

Proposition 3.6.3. *Let $1 \leq p < 2$. Then*

$$L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c) = L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c) + L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c) \quad \textit{with equivalent norms,}$$

where the sum is taken in $L_p(\mathcal{M}; \ell_2^c)$.

We now turn back to the discussion on complementation results. Note that for $1 \leq p < 2$, the spaces h_p^c and $h_p^{1_c}$ embed isometrically into $L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c)$ and $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c)$ respectively. Indeed, a martingale $x \in h_p^{1_c}$ can be written as $x = \mathcal{D}(ba) = \sum_n d_n(b_n a_n)$, with $ba \in L_p(\mathcal{M}; \ell_1^c)$. Then

$$d_n(b_n a_n) = \mathcal{E}_n(b_n a_n) - \mathcal{E}_{n-1}(b_n a_n) = \sum_k u_n(b_n^*)(k)^* u_n(a_n)(k) - \sum_k u_{n-1}(b_n^*)(k)^* u_{n-1}(a_n)(k)$$

is in $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c)$ for $u_n(a_n)(k), u_{n-1}(a_n)(k) \in L_q(\mathcal{M}_n)$. Moreover we have the following complementation result.

Proposition 3.6.4. *Let $1 \leq p < 2$. Then*

(i) The space h_p^c is complemented in $L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c)$.

(ii) The space $h_p^{1_c}$ is complemented in $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c)$.

Moreover, the projection is given in both cases by $\mathcal{D} : \sum_n e_{n,0} \otimes a_n \mapsto \sum_n d_n(a_n)$.

Proof. For (i), we simply use the fact that

$$\begin{aligned}\mathcal{E}_{n-1}|\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)|^2 &= \mathcal{E}_{n-1}|\mathcal{E}_n(a_n)|^2 - |\mathcal{E}_{n-1}(a_n)|^2 \\ &\leq \mathcal{E}_{n-1}|\mathcal{E}_n(a_n)|^2 \leq \mathcal{E}_{n-1}(\mathcal{E}_n|a_n|^2) = \mathcal{E}_{n-1}|a_n|^2.\end{aligned}$$

This shows that h_p^c is 1-complemented in $L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c)$. The second assertion was proved in Lemma 3.4.5 □

Remark 3.6.5. Observe that assertion (i) also holds true for $2 \leq p < \infty$.

We can now extend Lemma 3.6.1 to the case $p = 1$.

Proposition 3.6.6. *Let $1 \le p < \infty$. Then the space H_p^c is complemented in $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$.*

Proof. It suffices to consider $1 \le p < 2$. This result follows directly from Propositions 3.6.3 and 3.6.4. Indeed, let $a = (a_n)_n \in L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ be of norm < 1. By Proposition 3.6.3 there exist $b \in L_p^{\mathrm{cond}}(\mathcal{M}; \ell_1^c), c \in L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c)$ such that $a = b + c$ and

$$\|b\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)} + \|c\|_{L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)} \le C(p).$$

Then $\mathcal{D}(a) = \mathcal{D}(b) + \mathcal{D}(c)$ and Proposition 3.6.4 implies

$$\begin{aligned}\|\mathcal{D}(a)\|_{H_p^c} &\le \|\mathcal{D}(b)\|_{H_p^c} + \|\mathcal{D}(c)\|_{H_p^c} \le C'(p)\big(\|\mathcal{D}(b)\|_{h_p^{1_c}} + \|\mathcal{D}(c)\|_{h_p^c}\big)\\ &\le C''(p)\big(\|b\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)} + \|c\|_{L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)}\big) \le C(p)C''(p).\end{aligned}$$

Since all the constants involved here remain bounded as $p \to 1$, this ends the proof. □

Let us now describe a decomposition involving algebraic atoms. We first decompose the conditioned columns L_p-spaces for $1 \le p < 2$ by using a dual approach, then by the complementation results established above we will deduce an analogous decomposition for the Hardy spaces. Before defining the algebraic atoms for the conditioned column L_p-spaces, let us introduce $L_p^{\mathrm{ad}}(\mathcal{M}; \ell_2^c)$, the closed subspace of $L_p(\mathcal{M}; \ell_2^c)$ consisting of adapted columns for $1 \le p < \infty$, i.e.,

$$L_p^{\mathrm{ad}}(\mathcal{M}; \ell_2^c) = \Big\{ \sum_{n\ge 0} e_{n,0} \otimes a_n \ : \ a_n \in L_p(\mathcal{M}_n) \Big\} \subset L_p(\mathcal{M}; \ell_2^c).$$

We define the algebraic atoms associated to $L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c)$ and $L_p^{\mathrm{cond}-}(\mathcal{M}; \ell_2^c)$ respectively as follows.

Definition 3.6.7. *Let $1 \le p < 2$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$.*

1. *$x \in L_p(\mathcal{M}; \ell_2^c)$ is said to be an algebraic $L_p^{c,\mathrm{cond}}$-atom if we can write $x = ba$ where*

   (i) $b = \sum_{n \le k} e_{k,n} \otimes b_{k,n}$ is a lower triangular matrix in $L_2(B(\ell_2)\overline{\otimes}\mathcal{M})$ with $\|b\|_2 = \Big(\sum_{n\le k} \|b_{k,n}\|_2^2\Big)^{1/2} \le 1$;

   (ii) $a = \sum_{n\ge 0} e_{n,0} \otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M}; \ell_2^c)$ with $\|a\|_q = \Big\|\Big(\sum_{n\ge 0} |a_n|^2\Big)^{1/2}\Big\|_q \le 1.$

2. *$x \in L_p(\mathcal{M}; \ell_2^c)$ is said to be an algebraic $L_p^{c,\mathrm{cond}-}$-atom if we can write $x = ba$ where*

   *(i) $b = \sum_{n < k} e_{k,n} \otimes b_{k,n}$ is a **strictly** lower triangular matrix in $L_2(B(\ell_2)\overline{\otimes}\mathcal{M})$ with $\|b\|_2 = \Big(\sum_{n< k} \|b_{k,n}\|_2^2\Big)^{1/2} \le 1$;*

   (ii) $a = \sum_{n\ge 0} e_{n,0} \otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M}; \ell_2^c)$ with $\|a\|_q = \Big\|\Big(\sum_{n\ge 0} |a_n|^2\Big)^{1/2}\Big\|_q \le 1.$

We denote by $L_p^{c,\mathrm{cond,algat}}$ *(resp.* $L_p^{c,\mathrm{cond-,algat}}$*) the completion in* $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ *(resp.* $L_p^{\mathrm{cond-}}(\mathcal{M};\ell_2^c)$*) of the space whose unit ball is the absolute convex hull of algebraic* $L_p^{c,\mathrm{cond}}$*-atoms (resp. algebraic* $L_p^{c,\mathrm{cond-}}$*-atoms).*

Actually the sets of algebraic $L_p^{c,\mathrm{cond}}$-atoms and algebraic $L_p^{c,\mathrm{cond-}}$-atoms are already absolutely convex.

Lemma 3.6.8. *Let* $1 \leq p < 2$ *and* $(\lambda_m)_{1\leq m\leq M} \subset \mathbb{C}$ *be such that* $\sum_m |\lambda_m| \leq 1$*. For* $1 \leq m \leq M$*, let* $x^m \in L_p(\mathcal{M};\ell_2^c)$ *be an algebraic* $L_p^{c,\mathrm{cond}}$*-atom (resp. an algebraic* $L_p^{c,\mathrm{cond-}}$*-atom).*
Then $x = \sum_m \lambda_m x^m$ *is an algebraic* $L_p^{c,\mathrm{cond}}$*-atom (resp. algebraic* $L_p^{c,\mathrm{cond-}}$*-atom).*

Proof. We prove that the set of algebraic $L_p^{c,\mathrm{cond}}$-atoms is absolutely convex, the proof for the algebraic $L_p^{c,\mathrm{cond-}}$-atoms is similar. For each m we set

$$x^m = b^m a^m = \sum_{k\geq 0} e_{k,0} \otimes \Big(\sum_{n\leq k} b_{k,n}^m a_n^m\Big),$$

with

$$\Big(\sum_{n\leq k} \|b_{k,n}^m\|_2^2\Big)^{1/2} \leq 1 \quad \text{and} \quad \Big\|\Big(\sum_{n\geq 0} |a_n^m|^2\Big)^{1/2}\Big\|_q \leq 1.$$

Then we can write

$$x = \sum_{m=1}^M \lambda_m b^m a^m = b'a',$$

where

$$b' = \sum_{n\leq k} e_{k,n} \otimes b'_{k,n} \quad \text{and} \quad a' = \sum_{n\geq 0} e_{n,0} \otimes a'_n$$

are defined as follows. We first set

$$a'_n = \Big(\sum_{m=1}^M |\lambda_m||a_n^m|^2\Big)^{1/2}.$$

By approximation, we may assume that the a'_n's are invertible. Then we consider

$$v_n^m = \frac{\lambda_m}{\sqrt{|\lambda_m|}} a_n^m {a'}_n^{-1} \quad \text{and} \quad b'_{k,n} = \sum_m \sqrt{|\lambda_m|} b_{k,n}^m v_n^m.$$

It remains to see that b' and a' verify the required estimates. For b', since $\sum_m |v_n^m|^2 = 1$ for all n, we have by the Hölder inequality

$$\begin{aligned}
\|b'\|_2^2 = \sum_{n\leq k} \|{b'}_{k,n}\|_2^2 &= \sum_{n\leq k} \Big\|\sum_m \sqrt{|\lambda_m|} b_{k,n}^m v_n^m\Big\|_2^2 \\
&\leq \sum_{n\leq k} \Big\|\Big(\sum_m |\lambda_m| b_{k,n}^m (b_{k,n}^m)^*\Big)^{1/2}\Big\|_2^2 \Big\|\Big(\sum_m |v_n^m|^2\Big)^{1/2}\Big\|_\infty^2 \\
&= \sum_m |\lambda_m| \sum_{n\leq k} \|b_{k,n}^m\|_2^2 \leq 1.
\end{aligned}$$

The estimate for a' follows directly from the triangle inequality in $L_{q/2}(\mathcal{M})$

$$\begin{aligned}
\|a\|_q^2 &= \Big\|\Big(\sum_n \sum_m |\lambda_m||a_n^m|^2\Big)^{1/2}\Big\|_q^2 \\
&= \Big\|\sum_m |\lambda_m| \sum_n |a_n^m|^2\Big\|_{q/2} \leq \sum_m |\lambda_m| \|a^m\|_q^2 \leq 1.
\end{aligned}$$

□

Remark 3.6.9. As a consequence, we see that the unit ball of the algebraic atomic space $L_p^{c,\mathrm{cond},\mathrm{algat}}$ (resp. $L_p^{c,\mathrm{cond}-,\mathrm{algat}}$) is simply the completion in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ (resp. $L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)$) of the algebraic $L_p^{c,\mathrm{cond}}$-atoms (resp. algebraic $L_p^{c,\mathrm{cond}-}$-atoms).

We can now state the algebraic atomic decomposition for the conditioned column L_p-spaces as follows.

Theorem 3.6.10. *Let $1 \le p < 2$. Then*

(i) $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) = L_p^{c,\mathrm{cond},\mathrm{algat}}$,

(ii) $L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c) = L_p^{c,\mathrm{cond}-,\mathrm{algat}}$,

with equivalent norms.

Proof. We detail the proof of the first part, the second one being similar by a shift. Let

$$x = ba = \sum_{k\ge 0} e_{k,0} \otimes \Big(\sum_{n\le k} b_{k,n} a_n \Big)$$

be an algebraic $L_p^{c,\mathrm{cond}}$-atom. We want to estimate

$$\|x\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} = \Big\| \Big(\sum_{k\ge 0} \mathcal{E}_k |x_k|^2 \Big)^{1/2} \Big\|_p = \|v(x)\|_{L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))}$$

where $v : L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) \to L_p(N;\ell_2^c(\mathbb{N}^2))$ is the isometry given by

$$v(x) = \sum_{k\ge 0} e_{k,0} \otimes u_k(x_k) = \sum_{k,j\ge 0} e_{k,0} \otimes e_{j,0} \otimes u_k(x_k)(j).$$

Since u_k is a right $\mathcal{M}_k$-module map and $a_n \in L_q(\mathcal{M}_n) \subset L_q(\mathcal{M}_k)$ for all $n \le k$, we have

$$u_k(x_k) = u_k\Big(\sum_{n\le k} b_{k,n} a_n \Big) = u_k\Big(\sum_{n\le k} b_{k,n} \Big) a_n = \sum_{j\ge 0} e_{j,0} \otimes \sum_{n\le k} u_k(b_{k,n})(j) a_n.$$

Hence we can write

$$\begin{aligned} v(x) &= \sum_{k,j\ge 0} e_{k,0} \otimes e_{j,0} \otimes \sum_{n\le k} u_k(b_{k,n})(j) a_n \\ &= \Big(\sum_{n\le k, j\ge 0} e_{k,n} \otimes e_{j,0} \otimes u_k(b_{k,n})(j) \Big) \Big(\sum_{n\ge 0} e_{n,0} \otimes e_{0,0} \otimes a_n \Big). \end{aligned}$$

By the Hölder inequality this implies

$$\|v(x)\|_p \le \Big\| \sum_{n\le k, j\ge 0} e_{k,n} \otimes e_{j,0} \otimes u_k(b_{k,n})(j) \Big\|_2 \Big\| \sum_{n\ge 0} e_{n,0} \otimes e_{0,0} \otimes a_n \Big\|_q$$

for $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. The estimation of the b-term gives

$$\Big\| \sum_{n\le k} e_{k,n} \otimes u_k(b_{k,n}) \Big\|_2 = \Big(\sum_{n\le k} \tau(|u_k(b_{k,n})|^2) \Big)^{1/2} = \Big(\sum_{n\le k} \tau(\mathcal{E}_k |b_{k,n}|^2) \Big)^{1/2} = \Big(\sum_{n\le k} \|b_{k,n}\|_2^2 \Big)^{1/2}.$$

Thus we obtain

$$\|x\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} = \|v(x)\|_p \le \Big(\sum_{n\le k} \|b_{k,n}\|_2^2 \Big)^{1/2} \Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_q \le 1.$$

We use duality to prove the converse. Since the atomic space is defined as the completion in $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$, it suffices to show that the algebraic $L_p^{c,\mathrm{cond}}$-atoms are norming for $L_{p'}^{c,\mathrm{cond}}MO$. Indeed, if we show that the convex, bounded set D of algebraic $L_p^{c,\mathrm{cond}}$-atoms is C-norming for the dual of $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$, we will deduce that $B_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} \subset C\overline{D}^{\|\cdot\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)}}$. Let $x=(x_k)_k \in L_{p'}^{c,\mathrm{cond}}MO$. Then

$$\begin{aligned}
\|x\|_{L_{p'}^{c,\mathrm{cond}}MO} &= \sup\Big\{\Big(\sum_n \tau\Big(\mathcal{E}_n(\sum_{k\geq n}|x_k|^2)c_n\Big)\Big)^{1/2} : c_n \in L^+_{(p'/2)'}(\mathcal{M}),\ \Big\|\sum_n c_n\Big\|_{(p'/2)'} \leq 1\Big\} \\
&= \sup\Big\{\Big(\sum_{n\leq k} \|x_k(\mathcal{E}_n(c_n))^{1/2}\|_2^2\Big)^{1/2} : c_n \in L^+_{(p'/2)'}(\mathcal{M}),\ \Big\|\sum_n c_n\Big\|_{(p'/2)'} \leq 1\Big\} \\
&= \sup\Big\{\sum_{n\leq k} \tau(b_{k,n}^* x_k(\mathcal{E}_n(c_n))^{1/2}) : c_n \in L^+_{(p'/2)'}(\mathcal{M}),\ \Big\|\sum_n c_n\Big\|_{(p'/2)'} \leq 1,\ \sum_{n\leq k}\|b_{k,n}\|_2^2 \leq 1\Big\} \\
&= \sup\Big\{(x|ba) : c_n \in L^+_{(p'/2)'}(\mathcal{M}),\ \Big\|\sum_n c_n\Big\|_{(p'/2)'} \leq 1,\ \sum_{n\leq k}\|b_{k,n}\|_2^2 \leq 1\Big\},
\end{aligned}$$

where $b=\sum_{n\leq k} e_{k,n}\otimes b_{k,n}$ is a lower triangular matrix in L_2 and $a=\sum_{n\geq 0} e_{n,0}\otimes(\mathcal{E}_n(c_n))^{1/2}$ is in $L_q^{\mathrm{ad}}(\mathcal{M};\ell_2^c)$ of norm $\leq \delta'^{1/2}_{(p'/2)'} = \delta^{1/2}_{p'/2}$ (recall that the Doob constant $\delta_{p'/2}$ remains bounded as p' tends to ∞, i.e. as p tends to 1). Hence

$$\|x\|_{L_{p'}^{c,\mathrm{cond}}MO} \leq \delta^{1/2}_{p'/2} \sup_{y \text{ algebraic } L_p^{c,\mathrm{cond}}\text{-atom}} |(x|y)|.$$

□

It now suffices to use the Stein projection $\mathcal{D}$ to deduce the corresponding algebraic atomic decomposition of H_p^c and h_p^c.

Definition 3.6.11. *Let $1\leq p<2$. Then $x\in L_p(\mathcal{M})$ is said to be an algebraic H_p^c-atom (resp. algebraic h_p^c-atom) if x is of the form $x=\mathcal{D}(y)$ where y is an algebraic $L_p^{c,\mathrm{cond}}$-atom (resp. algebraic $L_p^{c,\mathrm{cond}-}$-atom).*
We denote by $H_p^{c,\mathrm{algat}}$ (resp. $h_p^{c,\mathrm{algat}}$) the completion in H_p^c (resp. h_p^c) of the space whose unit ball is the absolute convex hull of algebraic H_p^c-atoms (resp. algebraic h_p^c-atoms).

Remark 3.6.12. 1. Note that x is an algebraic h_p^c-atom if and only if we can write $x=\sum_{n\geq 0} b_n a_n$ where

(i) $(b_n)_{n\geq 0}\in \ell_2(L_2(\mathcal{M}))$ with $\mathcal{E}_n(b_n)=0$ for all $n\geq 0$ and $\sum_{n\geq 0}\|b_n\|_2^2\leq 1$;

(ii) $a=\sum_{n\geq 0} e_{n,0}\otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M};\ell_2^c)$ with $\Big\|\Big(\sum_{n\geq 0}|a_n|^2\Big)^{1/2}\Big\|_q \leq 1$.

Indeed, if x is an algebraic h_p^c-atom, then there exists an algebraic $L_p^{c,\mathrm{cond}-}$-atom $y=ba$ such that

$$x=\mathcal{D}(ba)=\sum_{k\geq 0} d_k\Big(\sum_{n<k} b_{k,n}a_n\Big)=\sum_{k\geq 0} d_k\Big(\sum_{n<k} b_{k,n}\Big)a_n,$$

for $a_n \in L_q(\mathcal{M}_n) \subset L_q(\mathcal{M}_{k-1})$ for all $n < k$. Hence we can write $x = \sum_n b'_n a'_n$ where $b'_n = \sum_{k>n} d_k(b_{k,n})$ and $a'_n = a_n$ satisfy the conditions (i) and (ii) above. Conversely, let $x = \sum_n b'_n a'_n$ be as above. Then

$$d_k(b'_n a'_n) = \begin{cases} d_k(\mathcal{E}_n(b'_n a'_n)) = d_k(\mathcal{E}_n(b'_n) a'_n) = 0 & \text{if} \quad k \leq n \\ d_k(b'_n) a'_n & \text{if} \quad k > n \end{cases}.$$

Thus

$$x = \sum_{k>n} d_k(b'_n) a'_n = \mathcal{D}(ba),$$

where $b = \sum_{n<k} e_{k,n} \otimes d_k(b'_n)$ and $a = \sum_{n\geq 0} e_{n,0} \otimes a'_n$ satisfy the conditions of an algebraic $L_p^{c,\mathrm{cond}-}$-atom.

2. Observe that by Lemma 3.6.8, the sets of algebraic H_p^c and h_p^c-atoms are also absolutely convex.

Then combining Propositions 3.6.4, 3.6.6 with Theorem 3.6.10 we get the following decomposition for the Hardy spaces.

Theorem 3.6.13. *Let $1 \leq p < 2$. Then*

(i) $H_p^c = H_p^{c,\mathrm{algat}}$,

(ii) $h_p^c = h_p^{c,\mathrm{algat}}$,

with equivalent norms.

Proof. We only detail the proof of (i), the second assertion being similar. It is clear that an algebraic H_p^c-atom is in H_p^c. Conversely, if $x \in H_p^c$ with $\|x\|_{H_p^c} \leq 1$, then $i(x)$ is in the unit ball of $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$. By Theorem 3.6.10, for $\varepsilon > 0$ there exists an algebraic $L_p^{c,\mathrm{cond}}$-atom y such that $\|i(x) - y\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} < \varepsilon$. Hence $\mathcal{D}(y)$ is an algebraic H_p^c-atom, and since $x = \mathcal{D}(i(x))$ we get by Proposition 3.6.6

$$\|x - \mathcal{D}(y)\|_{H_p^c} = \|\mathcal{D}(i(x) - y)\|_{H_p^c} \leq C(p)\|i(x) - y\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} < C(p)\varepsilon.$$

We conclude that x is in $H_p^{c,\mathrm{algat}}$. □

Remark 3.6.14. We can prove directly this Theorem by using the same argument as in Theorem 3.6.10. By this way we may obtain a better decomposition, in the sense that in the definition of algebraic atoms we can suppose that the $b_{k,n}$'s are in $L_2(\mathcal{M}_k)$.

Remark 3.6.15. Observe that the space $h_p^{1_c}$ is already defined using algebraic atoms similar to the algebraic H_p^c and h_p^c-atoms. Moreover, by Proposition 3.6.4 (ii) we see that the elements in $h_p^{1_c}$ are of the form $x = \mathcal{D}(y)$ with $y \in L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)$. We will say that $x \in L_p(\mathcal{M})$ is an algebraic $h_p^{1_c}$-atom if x is of the form $x = \mathcal{D}(y)$ where y is in the unit ball of $L_p^{\mathrm{cond}}(\mathcal{M};\ell_1^c)$, i.e., $y = ba$ with

(i) $b = \sum_n e_{n,n} \otimes b_n$ is a diagonal matrix in $L_2(\mathcal{M}\overline{\otimes}B(\ell_2))$
with $\|b\|_2 = \Big(\sum_{n\geq 0} \|b_n\|_2^2\Big)^{1/2} \leq 1$;

(ii) $a = \sum_{n\geq 0} e_{n,0} \otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M}; \ell_2^c)$ with $\|a\|_q = \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_q \leq 1$.

This describes naturally the space $h_p^{1_c}$ in the same way as the atomic spaces $H_p^{c,\mathrm{algat}}$ and $h_p^{c,\mathrm{algat}}$. Moreover, we could suppose in addition that b is an adapted diagonal, i.e., $b_n \in L_2(\mathcal{M}_n)$ for all n in the definition of an algebraic $h_p^{1_c}$-atom.

Our definitions of algebraic atoms for H_p^c, h_p^c and $h_p^{1_c}$ give an explicit Davis decomposition at the level of atoms for $1 \leq p < 2$. Observe that we can explicitly decompose an algebraic H_p^c-atom as follows. Let $x = \mathcal{D}(y)$ be such that y is an algebraic $L_p^{c,\mathrm{cond}}$-atom. Then $y = ba$ where $b = \sum_{n\leq k} e_{k,n} \otimes b_{k,n}$ is a lower triangular matrix of norm less than one in $L_2(\mathcal{M}\overline{\otimes}B(\ell_2))$ and $a = \sum_{n\geq 0} e_{n,0} \otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M}; \ell_2^c)$ is of norm less than one. Writing

$$b = \sum_{n<k} e_{k,n} \otimes b_{k,n} + \sum_{n\geq 0} e_{n,n} \otimes b_{n,n} =: b' + b'',$$

we get

$$x = \mathcal{D}(ba) = \mathcal{D}(b'a) + \mathcal{D}(b''a) =: x' + x'',$$

where x' is an algebraic h_p^c-atom and x'' an algebraic $h_p^{1_c}$-atom.

3.6.2 Decomposition into algebraic atoms of the ultraproducts of the conditioned column L_p-spaces

In the spirit of this paper, we start by considering the decomposition for the ultraproduct spaces. We will follow the approach detailed in subsection 3.6.1, by first looking at the conditioned column L_p-spaces.

Definition 3.6.16. *Let $1 \leq p < 2$. We define*

$$\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c(\sigma)) \quad and \quad K_p^{c,\mathrm{cond}}(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}}(\mathcal{U}))}^{\|\ \|_{\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})}},$$

where $I_{\widetilde{p},p} : \widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}}(\mathcal{U}) \to \widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ denotes the contractive ultraproduct of the componentwise inclusion maps. We define similarly the spaces

$$\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U}),\ K_p^{c,\mathrm{cond}-}(\mathcal{U}) \quad and \quad \widetilde{K}_p^{1_c,\mathrm{cond}}(\mathcal{U}),\ K_p^{1_c,\mathrm{cond}}(\mathcal{U}).$$

Observe that $\widetilde{\mathcal{H}}_p^c(\mathcal{U}), \widetilde{\mathsf{h}}_p^c(\mathcal{U})$ and $\widetilde{\mathsf{h}}_p^{1_c}(\mathcal{U})$ embed isometrically into $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$, $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$ and $\widetilde{K}_p^{1_c,\mathrm{cond}}(\mathcal{U})$ respectively via the map i. Since i preserves the regularized spaces, $\mathcal{H}_p^c(\mathcal{U}), \mathsf{h}_p^c(\mathcal{U})$ and $\mathsf{h}_p^{1_c}(\mathcal{U})$ are also isometrically embedded into $K_p^{c,\mathrm{cond}}(\mathcal{U})$, $K_p^{c,\mathrm{cond}-}(\mathcal{U})$ and $K_p^{1_c,\mathrm{cond}}(\mathcal{U})$ respectively. Moreover, by Propositions 3.6.4 and 3.6.6 the map $\mathcal{D}$ is bounded for $1 \leq p < 2$ on $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$, $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$ and $\widetilde{K}_p^{1_c,\mathrm{cond}}(\mathcal{U})$, and it also preserves the regularized spaces. Hence we have

Proposition 3.6.17. *Let $1 \leq p < 2$. Then*

(i) The space $\mathcal{H}_p^c(\mathcal{U})$ is complemented in $K_p^{c,\mathrm{cond}}(\mathcal{U})$.

(ii) The space $\mathsf{h}_p^c(\mathcal{U})$ is complemented in $K_p^{c,\mathrm{cond}-}(\mathcal{U})$.

(iii) The space $\mathsf{h}_p^{1_c}(\mathcal{U})$ is complemented in $K_p^{1_c,\mathrm{cond}}(\mathcal{U})$.

Moreover, the projection is given in all cases by $\mathcal{D} = (\mathcal{D}_\sigma)^\bullet$.

Before defining the algebraic atoms for the ultraproducts of the conditioned column L_p-spaces, let us introduce

$$\widetilde{K}_p^{c,\mathrm{ad}}(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p^{\mathrm{ad}}(\mathcal{M};\ell_2^c(\sigma)) \quad \text{and} \quad K_p^{c,\mathrm{ad}}(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{K}_{\widetilde{p}}^{c,\mathrm{ad}}(\mathcal{U}))}^{\|\ \|_{\widetilde{K}_p^{c,\mathrm{ad}}(\mathcal{U})}}.$$

Definition 3.6.18. *Let $1 \leq p < 2$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$.*

1. *$\xi \in \widetilde{K}_p^c(\mathcal{U})$ is said to be an algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atom if we can write $\xi = \beta\alpha$ where*
   (i) $\beta = (\beta_\sigma)^\bullet \in \prod_{\mathcal{U}} L_2(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})$, $\beta_\sigma = \sum_{t\leq s, s,t\in\sigma} e_{s,t} \otimes \beta_\sigma(s,t)$ is a lower triangular matrix in $L_2(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})$ with
   $$\|\beta\|_2 = \lim_{\sigma,\mathcal{U}} \Big(\sum_{t\leq s, s,t\in\sigma} \|\beta_\sigma(s,t)\|_2^2 \Big)^{1/2} \leq 1;$$
   (ii) $\alpha = (\alpha_\sigma)^\bullet \in \widetilde{K}_q^{c,\mathrm{ad}}(\mathcal{U})$, $\alpha_\sigma = \sum_{t\in\sigma} e_{t,0}\otimes\alpha_\sigma(t)$ is an adapted column in $L_q(\mathcal{M};\ell_2^c(\sigma))$ with
   $$\|\alpha\|_q = \lim_{\sigma,\mathcal{U}} \Big\| \Big(\sum_{t\in\sigma} |\alpha_\sigma(t)|^2 \Big)^{1/2} \Big\|_q \leq 1$$
2. *$\xi \in \widetilde{K}_p^c(\mathcal{U})$ is said to be an algebraic $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$-atom if we can write $\xi = \beta\alpha$ as before, where in this case β_σ is a **strictly** lower triangular matrix.*

We denote by $\widetilde{K}_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U})$ (resp. $\widetilde{K}_p^{c,\mathrm{cond}-,\mathrm{algat}}(\mathcal{U})$) the completion in $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ (resp. $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$) of the space whose unit ball is the absolute convex hull of algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atoms (resp. algebraic $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$-atoms).

Remark 3.6.19. In this definition we omitted to write some maps, in order to simplify the statement. Actually, in (i) for instance, we should write $\rho_p^{c,\mathrm{cond}}(\xi) \in \widetilde{K}_p^c(\mathcal{U})$, where $\rho_p^{c,\mathrm{cond}}$ denotes the ultraproduct map of the componentwise bounded maps $\rho_p^{c,\mathrm{cond}} : \widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U}) \to \widetilde{K}_p^c(\mathcal{U})$.

Remark 3.6.20. Note that $\xi = (\xi_\sigma)^\bullet \in \widetilde{K}_p^c(\mathcal{U})$ is an algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atom if and only if for each σ, $\xi_\sigma = \beta_\sigma\alpha_\sigma$ is an algebraic $L_p^{c,\mathrm{cond}}(\sigma)$-atom. Hence Lemma 3.6.8 yields that the set of algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atoms is already absolutely convex. The same holds true for the algebraic $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$-atoms. Moreover, we can see that

$$\widetilde{K}_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p^{c,\mathrm{cond},\mathrm{algat}}(\sigma).$$

Indeed, let ξ be in the unit ball of $\widetilde{K}_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U})$ and fix $\varepsilon > 0$. Then there exists an algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atom $\beta\alpha = (\beta_\sigma\alpha_\sigma)^\bullet$ such that

$$\|\xi - \beta\alpha\|_{\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})} = \lim_{\sigma,\mathcal{U}} \|\xi_\sigma - \beta_\sigma\alpha_\sigma\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))} < \varepsilon.$$

Since we may assume $\|\xi_\sigma - \beta_\sigma\alpha_\sigma\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))} < \varepsilon$ for all σ, and $\beta_\sigma\alpha_\sigma$ is an algebraic $L_p^{c,\mathrm{cond}}(\sigma)$-atom we deduce that ξ_σ is in the unit ball of $L_p^{c,\mathrm{cond},\mathrm{algat}}(\sigma)$, hence ξ is in the unit ball of $\prod_{\mathcal{U}} L_p^{c,\mathrm{cond},\mathrm{algat}}(\sigma)$. Conversely, let $\xi = (\xi_\sigma)^\bullet$ be in the unit ball of $\prod_{\mathcal{U}} L_p^{c,\mathrm{cond},\mathrm{algat}}(\sigma)$. We may assume that ξ_σ is in the unit ball of $L_p^{c,\mathrm{cond},\mathrm{algat}}(\sigma)$ for all σ. Hence for each σ, there exists an algebraic $L_p^{c,\mathrm{cond}}(\sigma)$-atom $\beta_\sigma\alpha_\sigma$ such that

$$\|\xi_\sigma - \beta_\sigma\alpha_\sigma\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))} < \varepsilon.$$

Then $\beta\alpha = (\beta_\sigma\alpha_\sigma)^\bullet$ is an algebraic $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$-atom and

$$\|\xi - \beta\alpha\|_{\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})} = \lim_{\sigma,\mathcal{U}} \|\xi_\sigma - \beta_\sigma\alpha_\sigma\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))} < \varepsilon.$$

Thus Theorem 3.6.10 yields directly the following decomposition of the ultraproduct of conditioned column L_p-spaces in the algebraic atoms defined above.

Proposition 3.6.21. *Let $1 \le p < 2$. Then*

(i) $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U}) = \widetilde{K}_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U})$,

(ii) $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U}) = \widetilde{K}_p^{c,\mathrm{cond}-,\mathrm{algat}}(\mathcal{U})$,

with equivalent norms.

We define the regularized atoms as follows.

Definition 3.6.22. *Let $1 \le p < 2$. We say that ξ is an algebraic $K_p^{c,\mathrm{cond}}(\mathcal{U})$-atom (resp. algebraic $K_p^{c,\mathrm{cond}-}(\mathcal{U})$-atom) if $\xi = I_{\widetilde{p},p}(\eta)$ for some $\widetilde{p} > p$ and η an algebraic $\widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}}(\mathcal{U})$-atom (resp. algebraic $\widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}-}(\mathcal{U})$-atom).*
We denote by $K_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U})$ (resp. $K_p^{c,\mathrm{cond}-,\mathrm{algat}}(\mathcal{U})$) the completion in $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ (resp. $\widetilde{K}_p^{c,\mathrm{cond}-}(\mathcal{U})$) of the space whose unit ball is the (absolute convex hull of) algebraic $K_p^{c,\mathrm{cond}}(\mathcal{U})$-atoms (resp. algebraic $K_p^{c,\mathrm{cond}-}(\mathcal{U})$-atoms).

Remark 3.6.23. This definition of the regularized algebraic atoms simply means that in Definition 3.6.18, we replace $\alpha \in \widetilde{K}_q^{c,\mathrm{ad}}(\mathcal{U})$ by $\alpha \in K_q^{c,\mathrm{ad}}(\mathcal{U})$.

Then we get the regularized version of Proposition 3.6.21.

Proposition 3.6.24. *Let $1 \le p < 2$. Then*

(i) $K_p^{c,\mathrm{cond}}(\mathcal{U}) = K_p^{c,\mathrm{cond},\mathrm{algat}}(\mathcal{U})$,

(ii) $K_p^{c,\mathrm{cond}-}(\mathcal{U}) = K_p^{c,\mathrm{cond}-,\mathrm{algat}}(\mathcal{U})$,

with equivalent norms.

Proof. As before, we only detail (i). It is clear by the definition of the regularized space $K_p^{c,\mathrm{cond}}(\mathcal{U})$ that an algebraic $K_p^{c,\mathrm{cond}}(\mathcal{U})$-atom is in $K_p^{c,\mathrm{cond}}(\mathcal{U})$. Conversely, we use our standard argument. Let $\xi \in K_p^{c,\mathrm{cond}}(\mathcal{U})$ be of norm < 1, then by density we may assume that $\xi = I_{\widetilde{p},p}(\eta)$ where $\eta \in \widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}}(\mathcal{U})$ for some $\widetilde{p} > p$. Then viewing $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ as a subspace of $\prod_{\mathcal{U}} L_p(B(\ell_2(\sigma \times \mathbb{N}))\overline{\otimes}\mathcal{M})$ we can write

$$\|\xi\|_{\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})} = \lim_{q \to p, p < q \le \widetilde{p}} \|I_{\widetilde{p},q}(\eta)\|_{\widetilde{K}_q^{c,\mathrm{cond}}(\mathcal{U})} < 1.$$

Hence, we may assume in addition that $\|\eta\|_{\widetilde{K}^{c,\mathrm{cond}}_{\widetilde{p}}(\mathcal{U})} < 1$. Then by Proposition 3.6.21, for $\varepsilon > 0$ there exists an algebraic $\widetilde{K}^{c,\mathrm{cond}}_{\widetilde{p}}(\mathcal{U})$-atom $\beta\alpha$ such that $\|\eta - \beta\alpha\|_{\widetilde{K}^{c,\mathrm{cond}}_{\widetilde{p}}(\mathcal{U})} < \varepsilon$. Thus $I_{\widetilde{p},p}(\beta\alpha)$ is an algebraic $K^{c,\mathrm{cond}}_p(\mathcal{U})$-atom and

$$\|\xi - I_{\widetilde{p},p}(\beta\alpha)\|_{\widetilde{K}^{c,\mathrm{cond}}_p(\mathcal{U})} = \|I_{\widetilde{p},p}(\eta) - I_{\widetilde{p},p}(\beta\alpha)\|_{\widetilde{K}^{c,\mathrm{cond}}_p(\mathcal{U})} \leq \|\eta - \beta\alpha\|_{\widetilde{K}^{c,\mathrm{cond}}_{\widetilde{p}}(\mathcal{U})} < \varepsilon.$$

This ends the proof. □

We end this subsection by introducing the following terminology for the diagonal atoms, consistent with the previous definitions of algebraic atoms for $K^{c,\mathrm{cond}}_p(\mathcal{U})$ and $K^{c,\mathrm{cond}-}_p(\mathcal{U})$. Let $1 \leq p < 2$. We will say that $\xi \in \widetilde{K}^c_p(\mathcal{U})$ is an algebraic $\widetilde{K}^{1_c,\mathrm{cond}}_p(\mathcal{U})$-atom if ξ is in the unit ball of $\widetilde{K}^{1_c,\mathrm{cond}}_p(\mathcal{U})$, i.e., we can write $\xi = \beta\alpha$ where

(i) $\beta = (\beta_\sigma)^\bullet \in \prod_{\mathcal{U}} \ell_2(\sigma; L_2(\mathcal{M}))$, $\beta_\sigma = \sum_{t\in\sigma} e_{t,t} \otimes \beta_\sigma(t)$ is a diagonal matrix in $L_2(B(\ell_2(\sigma))\overline{\otimes}\mathcal{M})$ with
$$\|\beta\|_2 = \lim_{\sigma,\mathcal{U}} \Big(\sum_{t\in\sigma} \|\beta_\sigma(t)\|_2^2\Big)^{1/2} \leq 1;$$

(ii) $\alpha = (\alpha_\sigma)^\bullet \in \widetilde{K}^{c,\mathrm{ad}}_q(\mathcal{U})$, $\alpha_\sigma = \sum_{t\in\sigma} e_{t,0} \otimes \alpha_\sigma(t)$ is an adapted column in $L_q(\mathcal{M}; \ell_2^c(\sigma))$ with
$$\|\alpha\|_q = \lim_{\sigma,\mathcal{U}} \Big\|\Big(\sum_{t\in\sigma} |\alpha_\sigma(t)|^2\Big)^{1/2}\Big\|_q \leq 1$$

We say that ξ is an algebraic $K^{1_c,\mathrm{cond}}_p(\mathcal{U})$-atom if $\xi = I_{\widetilde{p},p}(\eta)$ for some $\widetilde{p} > p$ and η an algebraic $\widetilde{K}^{1_c,\mathrm{cond}}_{\widetilde{p}}(\mathcal{U})$-atom.

3.6.3 Decomposition into algebraic atoms of the Hardy spaces

We now turn to the decomposition of the Hardy spaces, by applying the Stein projection $\mathcal{D} = (\mathcal{D}_\sigma)^\bullet$ to the conditioned algebraic L_p-atoms. We start with the decomposition of the regularized ultraproduct of Hardy spaces.

Definition 3.6.25. *Let $1 \leq p < 2$. We say that $x \in L_p(\mathcal{M}_\mathcal{U})$ is an algebraic $\mathcal{H}^c_p(\mathcal{U})$-atom (resp. algebraic $\mathsf{h}^c_p(\mathcal{U})$-atom, algebraic $\mathsf{h}^{1_c}_p(\mathcal{U})$-atom) if x is of the form $x = \mathcal{D}(\xi)$ where ξ is an algebraic $K^{c,\mathrm{cond}}_p(\mathcal{U})$-atom (resp. algebraic $K^{c,\mathrm{cond}-}_p(\mathcal{U})$-atom, algebraic $K^{1_c,\mathrm{cond}}_p(\mathcal{U})$-atom).*
We denote by $\mathcal{H}^{c,\mathrm{algat}}_p(\mathcal{U})$ (resp. $\mathsf{h}^{c,\mathrm{algat}}_p(\mathcal{U})$, $\mathsf{h}^{1_c,\mathrm{algat}}_p(\mathcal{U})$) the completion in $\widetilde{\mathcal{H}}^c_p(\mathcal{U})$ (resp. $\widetilde{\mathsf{h}}^c_p(\mathcal{U})$, $\widetilde{\mathsf{h}}^{1_c}_p(\mathcal{U})$) of the space whose unit ball is the (absolute convex hull of) algebraic $\mathcal{H}^c_p(\mathcal{U})$-atoms (resp. algebraic $\mathsf{h}^c_p(\mathcal{U})$-atoms, algebraic $\mathsf{h}^{1_c}_p(\mathcal{U})$-atoms).

Remark 3.6.26. 1. As in the definition of the algebraic $\widetilde{K}^c_p(\mathcal{U})$-atoms, we omitted to write some maps to simplify the statement. For instance, for an algebraic $\mathcal{H}^c_p(\mathcal{U})$-atom we should consider $J^c_p(x) \in L_p(\mathcal{M}_\mathcal{U})$, where $J^c_p : \mathcal{H}^c_p(\mathcal{U}) \to L_p(\mathcal{M}_\mathcal{U})$.

2. The set of algebraic $\mathcal{H}^c_p(\mathcal{U})$-atoms (resp. algebraic $\mathsf{h}^c_p(\mathcal{U})$-atoms , algebraic $\mathsf{h}^{1_c}_p(\mathcal{U})$-atoms) is already absolutely convex.

Combining Proposition 3.6.24 with Proposition 3.6.17, we get

Proposition 3.6.27. *Let $1 \leq p < 2$. Then*

(i) $\mathcal{H}_p^c(\mathcal{U}) = \mathcal{H}_p^{c,\mathrm{algat}}(\mathcal{U})$,

(ii) $\mathsf{h}_p^c(\mathcal{U}) = \mathsf{h}_p^{c,\mathrm{algat}}(\mathcal{U})$,

(ii) $\mathsf{h}_p^{1_c}(\mathcal{U}) = \mathsf{h}_p^{1_c,\mathrm{algat}}(\mathcal{U})$,

with equivalent norms.

We now define the algebraic atoms for the Hardy spaces.

Definition 3.6.28. *Let $1 \leq p < 2$.*

1. *$x \in L_2(\mathcal{M})$ is said to be an algebraic $\mathcal{H}_p^c$-atom if, for some partition σ_0, x is an algebraic $H_p^c(\sigma)$-atom for all partitions $\sigma \supset \sigma_0$.*

2. *$x \in L_2(\mathcal{M})$ is said to be an algebraic h_p^c-atom if x is an algebraic $h_p^c(\sigma)$-atom for some partition σ.*

3. *$x \in L_2(\mathcal{M})$ is said to be an algebraic $\mathsf{h}_p^{1_c}$-atom if x is an algebraic $h_p^{1_c}(\sigma)$-atom for all partitions σ.*

We denote by $\mathcal{H}_p^{c,\mathrm{algat}}$ (resp. $\mathsf{h}_p^{c,\mathrm{algat}}$, $\mathsf{h}_p^{1_c,\mathrm{algat}}$) the completion in $\mathcal{H}_p^c$ (resp. h_p^c, $\mathsf{h}_p^{1_c}$) of the space whose unit ball is the absolute convex hull of algebraic $\mathcal{H}_p^c$-atoms (resp. algebraic h_p^c-atoms, algebraic $\mathsf{h}_p^{1_c}$-atoms).

In fact our set of algebraic h_p^c-atoms is already absolutely convex.

Lemma 3.6.29. *Let $1 \leq p < 2$. Let $\sigma^1, \cdots, \sigma^M$ be partitions contained in some partition σ, let $(\lambda_m)_{1\leq m \leq M}$ be a sequence of complex numbers such that $\sum_m |\lambda_m| \leq 1$ and let $x^1, \cdots, x^M \in L_2(\mathcal{M})$ be such that x^m is an algebraic $h_p^c(\sigma^m)$-atom for each $1 \leq m \leq M$. Then*

$$x = \sum_{m=1}^{M} \lambda_m x^m \text{ is an } h_p^c(\sigma)\text{-atom.}$$

In particular, if $\sigma \subset \sigma'$ then every $h_p^c(\sigma)$-atom is an $h_p^c(\sigma')$-atom. Hence $x \in L_2(\mathcal{M})$ is an algebraic h_p^c-atom if and only if for some partition σ_0, x is an algebraic $h_p^c(\sigma)$-atom for all $\sigma \supset \sigma_0$.

Proof. Since x^m is an algebraic $h_p^c(\sigma^m)$-atom, there exists an algebraic $L_p^{c,\mathrm{cond}-}(\sigma)$-atom $b^m a^m$ such that $x^m = \mathcal{D}_{\sigma^m}(b^m a^m)$. We can write for all $s \in \sigma^m$

$$d_s^{\sigma^m}(x^m) = d_s^{\sigma^m}\Big(\sum_{t<s, t\in\sigma^m} b_{s,t}^m a_t^m\Big) = \sum_{t<s,t\in\sigma^m} d_s^{\sigma^m}(b_{s,t}^m) a_t^m.$$

For each $s \in \sigma$ we denote by $s_m(s)$ the unique element in σ^m such that $s_m(s)^- \leq s^- < s \leq s_m(s)$. Then for $s \in \sigma$ we obtain

$$d_s^\sigma(x) = \sum_m \sum_{t<s_m(s),t\in\sigma^m} \lambda_m \, d_s^\sigma(b_{s_m(s),t}^m) a_t^m.$$

Indeed, we have

$$\begin{aligned} d_s^\sigma(x) &= \sum_m \lambda_m d_s^\sigma(x^m) = \sum_m \lambda_m d_s^\sigma(d_{s_m(s)}^{\sigma^m}(x^m)) \\ &= \sum_m \lambda_m d_s^\sigma\Big(\sum_{t<s_m(s),t\in\sigma^m} d_{s_m(s)}^{\sigma^m}(b_{s_m(s),t}^m)a_t^m\Big) \\ &= \sum_m \lambda_m \sum_{t<s_m(s),t\in\sigma^m} d_s^\sigma(b_{s_m(s),t}^m)a_t^m. \end{aligned}$$

Then we can write

$$d_s^\sigma(x) = \sum_{t<s,t\in\sigma} d_s^\sigma(b_{s,t})a_t,$$

where $b_{s,t}$ and a_t are defined as follows. We first set for $t \in \sigma$

$$a_t = \Big(\sum_m |\lambda_m| \mathbb{1}(t\in\sigma^m)|a_t^m|^2\Big)^{1/2}.$$

By approximation, we may assume that the a_t's are invertible. Then we consider

$$v_t^m = \frac{\lambda_m}{\sqrt{|\lambda_m|}}\mathbb{1}(t\in\sigma^m)a_t^m a_t^{-1} \quad \text{and} \quad b_{s,t} = \sum_m \mathbb{1}(t<s_m(s))\sqrt{|\lambda_m|}b_{s_m(s),t}^m v_t^m.$$

Note that a_t and v_t^m are adapted, hence

$$d_s^\sigma(b_{s,t}) = \sum_m \mathbb{1}(t<s_m(s))\sqrt{|\lambda_m|}d_s^\sigma(b_{s_m(s),t}^m)v_t^m.$$

It remains to see that $b = \sum_{t<s,s,t\in\sigma} e_{s,t}\otimes b_{s,t}$ and $a = \sum_{t\in\sigma} e_{t,0}\otimes a_t$ verify the required estimates, then since $x \in L_2(\mathcal{M})$ we will deduce that x is an algebraic $h_p^c(\sigma)$-atom. For b, since $\sum_m |v_t^m|^2 = 1$ for all $t\in\sigma$, we have by the Hölder inequality

$$\begin{aligned} \|b\|_2^2 &= \sum_{t<s,s,t\in\sigma} \|b_{s,t}\|_2^2 = \sum_{t<s,s,t\in\sigma} \Big\|\sum_m \mathbb{1}(t<s_m(s))\mathbb{1}(t\in\sigma^m)\sqrt{|\lambda_m|}b_{s_m(s),t}^m v_t^m\Big\|_2^2 \\ &\leq \sum_{t<s\in\sigma,s,t\in\sigma} \Big\|\Big(\sum_m \mathbb{1}(t<s_m(s))\mathbb{1}(t\in\sigma^m)|\lambda_m|b_{s_m(s),t}^m(b_{s_m(s),t}^m)^*\Big)^{1/2}\Big\|_2^2 \Big\|\Big(\sum_m |v_t^m|^2\Big)^{1/2}\Big\|_\infty^2 \\ &= \sum_m |\lambda_m| \sum_{t<s_m(s),t\in\sigma^m} \|b_{s_m(s),t}^m\|_2^2 = \sum_m |\lambda_m|\|b^m\|_2^2 \leq 1. \end{aligned}$$

The estimate for a follows directly from the triangle inequality in $L_{q/2}(\mathcal{M})$

$$\begin{aligned} \|a\|_q^2 &= \Big\|\Big(\sum_{t\in\sigma}\sum_m |\lambda_m|\mathbb{1}(t\in\sigma^m)|a_t^m|^2\Big)^{1/2}\Big\|_q^2 \\ &= \Big\|\sum_m |\lambda_m| \sum_{t\in\sigma^m} |a_t^m|^2\Big\|_{q/2} \leq \sum_m |\lambda_m|\|a^m\|_q^2 \leq 1. \end{aligned}$$

□

Applying the conditional expectation $\mathcal{E}_\mathcal{U}$ to Proposition 3.6.27 we get

Theorem 3.6.30. *Let $1 \leq p < 2$. Then*

(i) $\mathsf{h}_p^{1_c} = \mathsf{h}_p^{1_c,\mathrm{algat}}$,

(ii) $\mathsf{h}_p^c = \mathsf{h}_p^{c,\mathrm{algat}}$,

(iii) $\mathcal{H}_p^c = \mathcal{H}_p^{c,\mathrm{algat}} = \mathsf{h}_p^{1_c,\mathrm{algat}} + \mathsf{h}_p^{c,\mathrm{algat}}$,

with equivalent norms.

Proof. Assertion (i) follows from Lemma 3.4.9 and Remark 3.6.15.

For (ii), we first show that the conditional expectation $\mathcal{E}_{\mathcal{U}}$ is bounded from $\mathsf{h}_p^{c,\mathrm{algat}}(\mathcal{U})$ to $\mathsf{h}_p^{c,\mathrm{algat}}$. Then by Proposition 3.3.28 and Proposition 3.6.27 we will deduce that

$$\mathsf{h}_p^c = \mathcal{E}_{\mathcal{U}}(\mathsf{h}_p^c(\mathcal{U})) = \mathcal{E}_{\mathcal{U}}(\mathsf{h}_p^{c,\mathrm{algat}}(\mathcal{U})) = \mathsf{h}_p^{c,\mathrm{algat}}$$

with equivalent norms. Let $x = (x_\sigma)^\bullet$ be in the unit ball of $\mathsf{h}_p^{c,\mathrm{algat}}(\mathcal{U})$. By density it suffices to consider that x is an algebraic $\mathsf{h}_p^c(\mathcal{U})$-atom, i.e., $x = \mathcal{D}(\xi)$ where ξ is an algebraic $K_p^{c,\mathrm{cond}-}(\mathcal{U})$-atom. Hence there exist $\widetilde{p} > p$ and an algebraic $\widetilde{K}_{\widetilde{p}}^{c,\mathrm{cond}-}(\mathcal{U})$-atom $\beta\alpha$ such that $x = \mathcal{D}(I_{\widetilde{p},p}(\beta\alpha))$. Note that $\alpha \in \widetilde{K}_{\widetilde{q}}^{c,\mathrm{ad}}(\mathcal{U})$ for $\frac{1}{\widetilde{p}} = \frac{1}{2} + \frac{1}{\widetilde{q}}$, thus for $q < \widehat{q} < \widetilde{q}$ we have $\alpha' = I_{\widetilde{q},\widehat{q}}(\alpha) \in K_{\widehat{q}}^{c,\mathrm{ad}}(\mathcal{U})$ and $I_{\widetilde{q},q}(\alpha) = I_{\widehat{q},q}(\alpha')$. Then we can write for $\frac{1}{\widehat{p}} = \frac{1}{2} + \frac{1}{\widehat{q}}$

$$x = \mathcal{D}(I_{\widetilde{p},p}(\beta\alpha)) = \mathcal{D}(\beta I_{\widetilde{q},q}(\alpha)) = \mathcal{D}(\beta I_{\widehat{q},q}(\alpha')) = \mathcal{D}(I_{\widehat{p},p}(\beta\alpha')),$$

with $\alpha' \in K_{\widehat{q}}^{c,\mathrm{ad}}(\mathcal{U})$. In particular we have $\alpha' \in K_{\widehat{q}}^c(\mathcal{U})$, and for $\varepsilon > 0$ fixed, by Lemma 3.2.17 there exists $\alpha'' \in K_\infty^c(\mathcal{U})$ such that $\|\alpha' - \alpha''\|_{\widehat{q}} < \varepsilon$. We have seen in the proof of Lemma 3.4.21 that we may assume $\alpha'' \in \prod_{\mathcal{U}} L_\infty(\mathcal{M}; \ell_2^c(\sigma))$. We set

$$\widetilde{\alpha} = \mathcal{E}(\alpha'') \quad \text{and} \quad \widetilde{\xi} = I_{\widehat{p},p}(\beta\widetilde{\alpha}),$$

where $\mathcal{E} = (\mathcal{E}_\sigma)^\bullet$ denotes the ultraproduct map of the discrete Stein projections

$$\mathcal{E}_\sigma\Big(\sum_{t\in\sigma} e_{t,0} \otimes a_\sigma(t)\Big) = \sum_{t\in\sigma} e_{t,0} \otimes \mathcal{E}_t(a_\sigma(t)).$$

Let us show that $\mathcal{E}_{\mathcal{U}}(\mathcal{D}(\widetilde{\xi})) \in \mathsf{h}_p^{c,\mathrm{algat}}$. For each σ, $\widetilde{x}_\sigma = \mathcal{D}_\sigma(\beta_\sigma\widetilde{\alpha}_\sigma)$ is an algebraic $h_{\widehat{p}}^c(\sigma)$-atom (up to a constant which does not depend on σ). Indeed, $\widetilde{\alpha}_\sigma$ is an adapted column and

$$\|\widetilde{\alpha}_\sigma\|_{L_{\widehat{q}}(\mathcal{M};\ell_2^c(\sigma))} = \|\mathcal{E}_\sigma(\alpha''_\sigma)\|_{L_{\widehat{q}}(\mathcal{M};\ell_2^c(\sigma))} \leq \|\mathcal{E}_\sigma : L_{\widehat{q}}(\mathcal{M};\ell_2^c(\sigma)) \to L_{\widehat{q}}^{\mathrm{ad}}(\mathcal{M};\ell_2^c(\sigma))\|\|\alpha''_\sigma\|_{\widehat{q}} \leq \gamma_{\widehat{q}}\|\alpha''_\sigma\|_\infty.$$

Moreover, $\mathcal{D}_\sigma(\beta_\sigma\widetilde{\alpha}_\sigma) \in L_2$, with a bad constant depending on σ. Indeed

$$\|\mathcal{D}_\sigma(\beta_\sigma\widetilde{\alpha}_\sigma)\|_2 \leq \sqrt{2}\|\beta_\sigma\widetilde{\alpha}_\sigma\|_2 \leq \sqrt{2}\|\beta_\sigma\|_2\|\widetilde{\alpha}_\sigma\|_\infty < \infty,$$

for

$$\begin{aligned}\|\widetilde{\alpha}_\sigma\|_\infty^2 = \|\mathcal{E}_\sigma(\alpha''_\sigma)\|_\infty^2 &= \Big\|\sum_{t\in\sigma}|\mathcal{E}_t(\alpha''_\sigma(t))|^2\Big\|_\infty \\ &\leq |\sigma|\sup_{t\in\sigma}\||\mathcal{E}_t(\alpha''_\sigma(t))|^2\|_\infty \leq |\sigma|\sup_{t\in\sigma}\|\alpha''_\sigma(t)\|_\infty^2 \leq |\sigma|\|\alpha''_\sigma\|_\infty.\end{aligned}$$

In particular $\widetilde{x}_\sigma$ is an algebraic h_p^c-atom. Hence we have $\mathcal{E}_{\mathcal{U}}(\mathcal{D}(\widetilde{\xi})) = w\text{-}\lim_{\sigma,\mathcal{U}}\widetilde{x}_\sigma$ in $L_{\widehat{p}}(\mathcal{M})$ and for all σ, $\widetilde{x}_\sigma \in h_{\widehat{p}}^c(\sigma) \subset \mathsf{h}_{\widehat{p}}^c$. Indeed, by the density of $L_2(\mathcal{M})$ in $h_{\widehat{p}}^c(\sigma)$, for

$\varepsilon > 0$ there exists $a_\sigma \in L_2(\mathcal{M})$ such that $\|\widetilde{x}_\sigma - a_\sigma\|_{h^c_{\widehat{p}}(\sigma)} < \varepsilon$. By Lemma 3.3.11 we get $\|\widetilde{x}_\sigma - a_\sigma\|_{\mathsf{h}^c_{\widehat{p}}} < 2^{1/\widehat{p}}\varepsilon$, hence $\widetilde{x}_\sigma \in \mathsf{h}^c_{\widehat{p}}$. Furthermore we have

$$\|\widetilde{x}_\sigma\|_{\mathsf{h}^c_{\widehat{p}}} \leq 2^{1/\widehat{p}}\|x_\sigma\|_{h^c_{\widehat{p}}(\sigma)} \leq 2^{1/\widehat{p}}\|\beta_\sigma\|_2\|\alpha_\sigma\|_{\widehat{q}} \leq 2^{1/\widehat{p}},$$

and the family $(\widetilde{x}_\sigma)_\sigma$ is uniformly bounded in $\mathsf{h}^c_{\widehat{p}}$. Since $\widehat{p} > 1$, the space $\mathsf{h}^c_{\widehat{p}}$ is reflexive and the weak-limit of the x_σ's exists in $\mathsf{h}^c_{\widehat{p}}$. Then taking convex combinations of the $\widetilde{x}_\sigma$'s, Lemma 3.6.29 ensures that we still have algebraic h^c_p-atoms and we obtain that $\mathcal{E}_\mathcal{U}(\mathcal{D}(\widetilde{\xi}))$ is in the closure in $\mathsf{h}^c_{\widehat{p}}$-norm of algebraic h^c_p-atoms, and hence also in the closure in h^c_p-norm. This shows that $\mathcal{E}_\mathcal{U}(\mathcal{D}(\widetilde{\xi})) \in \mathsf{h}^{c,\mathrm{algat}}_p$. It remains to show that

$$\|\mathcal{E}_\mathcal{U}(x) - \mathcal{E}_\mathcal{U}(\mathcal{D}(\widetilde{\xi}))\|_{\mathsf{h}^c_p} < C\varepsilon,$$

then we will deduce that $\mathcal{E}_\mathcal{U}(x) \in \mathsf{h}^{c,\mathrm{algat}}_p$. By Proposition 3.3.28 we have

$$\begin{aligned}
\|\mathcal{E}_\mathcal{U}(x) - \mathcal{E}_\mathcal{U}(\mathcal{D}(\widetilde{\xi}))\|_{\mathsf{h}^c_p} &\leq 2^{1/p}\|x - \mathcal{D}(\widetilde{\xi})\|_{\mathsf{h}^c_p(\mathcal{U})} \\
&= 2^{1/p}\|\mathcal{D}(I_{\widehat{p},p}(\beta\alpha')) - \mathcal{D}(I_{\widehat{p},p}(\beta\widetilde{\alpha}))\|_{\mathsf{h}^c_p(\mathcal{U})} \\
&= 2^{1/p}\|I_{\widehat{p},p}(\mathcal{D}(\beta(\alpha' - \widetilde{\alpha})))\|_{\mathsf{h}^c_p(\mathcal{U})} \\
&\leq 2^{1/p}\|\mathcal{D}(\beta(\alpha' - \widetilde{\alpha}))\|_{\mathsf{h}^c_{\widehat{p}}(\mathcal{U})} \leq 2^{1/p}\sqrt{2}\gamma_{\widehat{p}}\|\beta\|_2\|\alpha' - \widetilde{\alpha}\|_{\widehat{q}} \\
&\leq 2^{1/p}\sqrt{2}\gamma_{\widehat{p}}\|\alpha' - \mathcal{E}(\alpha'')\|_{\widehat{q}} = 2^{1/p}\sqrt{2}\gamma_{\widehat{p}}\|\mathcal{E}(\alpha' - \alpha'')\|_{\widehat{q}} \\
&\leq 2^{1/p}\sqrt{2}\gamma_{\widehat{p}}\gamma_{\widehat{q}}\|\alpha' - \alpha''\|_{\widehat{q}} \leq 2^{1/p}\sqrt{2}\gamma_{\widehat{p}}\gamma_{\widehat{q}}\varepsilon.
\end{aligned}$$

This concludes the proof of (ii). We need the Davis decomposition proved in Theorem 3.4.20 to deduce (iii). Indeed, we can show that

$$\mathcal{H}^{c,\mathrm{algat}}_p \overset{(1)}{\subset} \mathcal{H}^c_p \overset{(2)}{=} \mathsf{h}^{1_c}_p + \mathsf{h}^c_p \overset{(3)}{=} \mathsf{h}^{1_c,\mathrm{algat}}_p + \mathsf{h}^{c,\mathrm{algat}}_p \overset{(4)}{\subset} \mathcal{H}^{c,\mathrm{algat}}_p.$$

The inclusion (1) is obvious since an algebraic $\mathcal{H}^c_p$-atom is in $L_2(\mathcal{M})$. The equality (2) comes from Theorem 3.4.20, and (3) follows from the first part of this proof. Finally, it is clear that an algebraic $\mathsf{h}^{1_c}_p$-atom is also an algebraic $\mathcal{H}^c_p$-atom, and by Lemma 3.6.29 this also holds true for an algebraic h^c_p-atom. □

3.7 Interpolation

In this section we establish the expected interpolation results for the Hardy spaces associated to a continuous filtration. We will deal with the complex method of interpolation, and we refer to [2] for informations on interpolation. The interpolation results have already been used in the literature and are particularly important in abstract semigroup theory. The conditioned column L_p-spaces and the algebraic atomic decomposition introduced in the previous section will play a crucial role in the proof of these interpolation results.

3.7.1 The discrete case

Let us first describe the approach we will use to deal with interpolation of noncommutative Hardy spaces in the discrete setting. Let $(\mathcal{M}_n)_{n\geq 0}$ be a discrete filtration. We are dealing with the interpolation result

$$H_p^c = [BMO^c, H_1^c]_{\frac{1}{p}} \quad \text{for } 1 < p < \infty, \tag{3.7.1}$$

proved in [31]. See also [22] for a different proof with better constants. However, we will not use the same approach to extend this result to the continuous setting. Our method is based on the crucial observation stated in Proposition 3.6.6. The first step is to show that the conditioned column L_p-spaces $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ form an interpolation scale for $1 \leq p < \infty$. We will use the algebraic atoms introduced in subsection 3.6.1. Then by complementation we will deduce that the Hardy spaces H_p^c also form an interpolation scale for $1 \leq p < \infty$. Finally, we will use the standard duality argument to obtain the interpolation of the spaces $L_{p'}^c MO$ for $2 < p' \leq \infty$. Since $L_{p'}^c MO = H_{p'}^c$ for $2 < p' < \infty$, an application of the Wolff Theorem (see [51]) will yield the desired interpolation result (3.7.1).

Proposition 3.7.1. *Let $1 \leq p_1 < p_2 < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) = [L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c)]_\theta \quad \textit{with equivalent norms.}$$

Proof. Note that since $\mathcal{M}$ is finite, $L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c) \subset L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ and the couple $(L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c))$ is compatible. Recall that we can see $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ as a complemented subspace of $L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ for $1 < p < \infty$. Indeed, $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ is isometrically isomorphic to $L_p^{\mathrm{ad}}(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ via the map v, which is complemented in $L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ for $1 < p < \infty$ via the Stein projection

$$\mathcal{E} : \left\{ \begin{array}{ccc} L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2)) & \longrightarrow & L_p^{\mathrm{ad}}(\mathcal{M};\ell_2^c(\mathbb{N}^2)) \\ \displaystyle\sum_{n,k\geq 0} e_{n,0}\otimes e_{k,0}\otimes a_{n,k} & \longmapsto & \displaystyle\sum_{n,k\geq 0} e_{n,0}\otimes e_{k,0}\otimes \mathcal{E}_n(a_{n,k}) \end{array} \right. .$$

Since

$$L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2)) = [L_{p_1}(\mathcal{M};\ell_2^c(\mathbb{N}^2)), L_{p_2}(\mathcal{M};\ell_2^c(\mathbb{N}^2))]_\theta \tag{3.7.2}$$

holds isometrically for $1 \leq p_1 < p_2 \leq \infty$, we get

$$L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) = [L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c)]_\theta$$

with equivalent norms for $1 < p_1 < p_2 < \infty$. Now we want to allow $p_1 = 1$, and we may assume $1 \leq p_1 < p < p_2 < 2$. Indeed, an application of the Wolff Theorem will yield the general case $1 \leq p_1 < p_2 < \infty$. Let $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Recall that by Theorem 3.6.10 we have $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) = L_p^{c,\mathrm{cond,algat}}$. We consider an algebraic $L_p^{c,\mathrm{cond}}$-atom $x = ba$, with

$$b = \sum_{n\leq k} e_{k,n}\otimes b_{k,n} \quad , \quad a = \sum_{n\geq 0} e_{n,0}\otimes a_n \in L_q^{\mathrm{ad}}(\mathcal{M};\ell_2^c) \quad \text{and} \quad \|b\|_2 \leq 1,\ \|a\|_q \leq 1,$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. We set $2 \leq q_1 \leq q \leq q_2 < \infty$ such that $\frac{1}{p_j} = \frac{1}{2} + \frac{1}{q_j}$ $(j = 1, 2)$. We know that $L_q^{\mathrm{ad}}(\mathcal{M};\ell_2^c)$ is complemented in $L_q(\mathcal{M};\ell_2^c)$ for $1 < q < \infty$ via the Stein projection. Hence, since θ satisfies $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, the interpolation of the column L_p-spaces yields

$$L_q^{\mathrm{ad}}(\mathcal{M};\ell_2^c) = [L_{q_1}^{\mathrm{ad}}(\mathcal{M};\ell_2^c), L_{q_2}^{\mathrm{ad}}(\mathcal{M};\ell_2^c)]_\theta$$

with equivalent norms. Thus there exists $f \in \mathcal{F}(L_{q_1}^{\mathrm{ad}}(\mathcal{M};\ell_2^c), L_{q_2}^{\mathrm{ad}}(\mathcal{M};\ell_2^c))$ such that $f(\theta) = a$. Recall that $f \in \mathcal{F}(L_{q_1}^{\mathrm{ad}}(\mathcal{M};\ell_2^c), L_{q_2}^{\mathrm{ad}}(\mathcal{M};\ell_2^c))$ if f is continuous on the strip $S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$, analytic on $S = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ and satisfy

$$\|f(k+it)\|_{L_{q_k}^{\mathrm{ad}}(\mathcal{M};\ell_2^c)} \leq c_q, \ \lim_{|t|\to\infty} \|f(k+it)\|_{L_{q_k}^{\mathrm{ad}}(\mathcal{M};\ell_2^c)} = 0 \quad \text{for } k = 1, 2.$$

Then by setting

$$g(z) = bf(z) \quad \text{for} \ \ z \in S$$

we obtain an analytic function $g \in \mathcal{F}(L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c))$ such that $g(\theta) = x$. By density, this proves the inclusion

$$L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c) \subset [L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c)]_\theta.$$

The reverse inclusion is trivial by complementation. Indeed, $L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)$ is isomorphic to a subspace of $L_p(\mathcal{M};\ell_2^c(\mathbb{N}^2))$ for $1 \leq p < \infty$, which is complemented for $1 < p < \infty$. □

The complementation result proved in Proposition 3.6.6 then implies the

Corollary 3.7.2. *Let $1 \leq p_1 < p_2 < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$H_p^c = [H_{p_1}^c, H_{p_2}^c]_\theta \quad \textit{with equivalent norms.}$$

Using standard arguments we deduce

Theorem 3.7.3 ([31])**.** *Let $1 < p < \infty$. Then*

$$H_p^c = [BMO^c, H_1^c]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

Note that this approach also works for the conditioned Hardy spaces h_p^c, by considering $L_p^{\mathrm{cond}-}(\mathcal{M};\ell_2^c)$. Hence we recover the following result from Chapter 2

Theorem 3.7.4. *Let $1 < p < \infty$. Then*

$$h_p^c = [bmo^c, h_1^c]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

Recall that in Chapter, the proof of Theorem 3.7.4 relies on the following characterization of h_p^c for $1 \leq p \leq 2$

$$\|x\|_{h_p^c} \simeq N_p^c(x) \quad \text{for } x \in L_2(\mathcal{M}). \tag{3.7.3}$$

Here we set

$$N_p^c(x) = \inf_W \Big[\tau\Big(\sum_{n\geq -1} w_n^{1-2/p}|d_{n+1}(x)|^2\Big)\Big]^{1/2},$$

where W denotes the set of sequences $\{w_n\}_{n\geq -1}$ such that $\{w_n^{2/p-1}\}_{n\geq -1}$ is nondecreasing with each $w_n \in L_1^+(\mathcal{M}_n)$ invertible with bounded inverse and $\|w_n\|_1 \leq 1$. This characterization was originally introduced by Herz in [18] in the classical case. Moreover, we can deduce Theorem 3.7.3 from Theorem 3.7.4 by using the Davis decomposition and the interpolation of the diagonal Hardy spaces h_p^d. Since in our approach of the Davis decomposition in the continuous setting in Section 3.4 we used the spaces $h_p^{1_c}$ for the diagonal part, it may be interesting to have an interpolation result involving these spaces.

Proposition 3.7.5. *Let $0 < \theta < 1$.*

(i) Let $1 \leq p_1 < p < p_2 < 2$ *be such that* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$L_p(\mathcal{M}; \ell_1^c) = [L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c)]_\theta \quad \textit{isometrically.}$$

(ii) Let $2 \leq p_1 < p < p_2 < \infty$ *be such that* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$L_p(\mathcal{M}; \ell_\infty^c) = [L_{p_1}(\mathcal{M}; \ell_\infty^c), L_{p_2}(\mathcal{M}; \ell_\infty^c)]_\theta \quad \textit{isometrically.}$$

Proof. Assertion (ii) was proved in Proposition 3.7 of [31]. For (i), the inclusion

$$L_p(\mathcal{M}; \ell_1^c) \subset [L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c)]_\theta$$

can be proved by using the same argument as in the first part of the proof of Proposition 3.7.1. For the reverse inclusion we will show the dual version

$$L_{p'}(\mathcal{M}; \ell_\infty^c) = (L_p(\mathcal{M}; \ell_1^c))^* \subset ([L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c)]_\theta)^*. \tag{3.7.4}$$

Let $x \in L_{p'}(\mathcal{M}; \ell_\infty^c)$ be of norm < 1 and $y \in [L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c)]_\theta$. Fix $\varepsilon > 0$. Then there exists an analytic function $f \in \mathcal{F}(L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c))$ of norm

$$\|f\|_{\mathcal{F}(L_{p_1}(\mathcal{M};\ell_1^c),L_{p_2}(\mathcal{M};\ell_1^c))} = \max\big(\sup_t \|f(it)\|_{L_{p_1}(\mathcal{M};\ell_1^c)}\ ,\ \sup_t \|f(1+it)\|_{L_{p_2}(\mathcal{M};\ell_1^c)}\big) \leq \|y\|_\theta + \varepsilon$$

such that $f(\theta) = y$. On the other hand, by (ii) there exists an analytic function $g \in \mathcal{F}(L_{p_1'}(\mathcal{M}; \ell_\infty^c), L_{p_2'}(\mathcal{M}; \ell_\infty^c))$ of norm

$$\|g\|_{\mathcal{F}(L_{p_1'}(\mathcal{M};\ell_\infty^c),L_{p_2'}(\mathcal{M};\ell_\infty^c))} = \max\big(\sup_t \|g(it)\|_{L_{p_1'}(\mathcal{M};\ell_\infty^c)}\ ,\ \sup_t \|g(1+it)\|_{L_{p_2'}(\mathcal{M};\ell_\infty^c)}\big) < 1$$

such that $g(\theta) = x$. Setting

$$h(z) = \tau(g(\overline{z})^* f(z)) \quad \text{for} \quad z \in S$$

we get a continuous function on the strip S, analytic on the interior of S, satisfying the following estimates for $t \in \mathbb{R}$ and $k = 1, 2$

$$|h(k+it)| \leq \|g(k+it)\|_{L_{p_k'}(\mathcal{M};\ell_\infty^c)} \|f(k+it)\|_{L_{p_k}(\mathcal{M};\ell_1^c)} \leq \|y\|_\theta + \varepsilon.$$

Thus the three-lines Theorem implies that

$$|\tau(x^* y)| = |h(\theta)| \leq \|y\|_\theta + \varepsilon.$$

Sending ε to 0 gives $x \in ([L_{p_1}(\mathcal{M}; \ell_1^c), L_{p_2}(\mathcal{M}; \ell_1^c)]_\theta)^*$. This shows (3.7.4) and ends the proof of (i). ☐

Lemma 3.4.5 implies by complementation the similar result for the spaces $h_p^{1_c}$ and $h_p^{\infty_c}$.

Theorem 3.7.6. *Let* $0 < \theta < 1$.

(i) Let $1 \leq p_1 < p < p_2 < 2$ *be such that* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$h_p^{1_c} = [h_{p_1}^{1_c}, h_{p_2}^{1_c}]_\theta \quad \textit{with equivalent norms.}$$

(ii) Let $2 \leq p_1 < p < p_2 < \infty$ *be such that* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$h_p^{\infty_c} = [h_{p_1}^{\infty_c}, h_{p_2}^{\infty_c}]_\theta \quad \textit{with equivalent norms.}$$

Since Theorem 3.7.3 also holds true for the row spaces, by using the Burkholder-Gundy inequalities and Fefferman Stein duality we obtain the following interpolation result for the Hardy space H_p.

Theorem 3.7.7 ([31])**.** *Let* $1 < p < \infty$. *Then*

$$H_p = [BMO, H_1]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

3.7.2 Interpolation of the ultraproduct of the conditioned column L_p-spaces

We follow the approach detailed in the previous subsection and start by proving the following interpolation result for the spaces $K_p^{c,\mathrm{cond}}(\mathcal{U})$. Note that $(K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U}))$ is a compatible couple. Indeed, we can show that for any $1 < p < \infty$ the map $I_{p,1} : K_p^{c,\mathrm{cond}}(\mathcal{U}) \to K_1^{c,\mathrm{cond}}(\mathcal{U})$ is injective. Let us consider $V = (v_\sigma)^\bullet$, the ultraproduct map of the isometric inclusions

$$v_\sigma : \begin{cases} L_p^{\mathrm{cond}}(\mathcal{M}; \ell_2^c(\sigma)) & \longrightarrow & L_p^{\mathrm{ad}}(\mathcal{M}; \ell_2^c(\sigma \times \mathbb{N})) \\ \displaystyle\sum_{t\in\sigma} e_{t,0} \otimes a_t & \longmapsto & \displaystyle\sum_{t\in\sigma} e_{t,0} \otimes u_t(a_t) \end{cases} .$$

Then V sends isometrically $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ into $\widetilde{\mathsf{k}}_p^c(\mathcal{U})$, and it preserves the regularized spaces. Thus we have an isometric embedding

$$V : K_p^{c,\mathrm{cond}}(\mathcal{U}) \to \mathsf{k}_p^c(\mathcal{U}).$$

Since $\mathsf{k}_p^c(\mathcal{U}) \subset \mathsf{k}_1^c(\mathcal{U})$ injectively, we deduce that the inclusion $K_p^{c,\mathrm{cond}}(\mathcal{U}) \subset K_1^{c,\mathrm{cond}}(\mathcal{U})$ is also injective.

Proposition 3.7.8. *Let $1 \le p_1 < p_2 < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then*

$$K_p^{c,\mathrm{cond}}(\mathcal{U}) = [K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})]_\theta \quad \textit{with equivalent norms.}$$

Proof of Proposition 3.7.8. Let us first observe that $K_p^{c,\mathrm{cond}}(\mathcal{U})$ can be seen as a complemented subspace of $\mathsf{k}_p^c(\mathcal{U})$ for $1 < p < \infty$. Indeed, we define the spaces

$$\widetilde{\mathsf{k}}_p^{c,\mathrm{ad}}(\mathcal{U}) = \prod\nolimits_{\mathcal{U}} L_p^{\mathrm{ad}}(\mathcal{M}; \ell_2^c(\sigma \times \mathbb{N})) \quad \text{and} \quad \mathsf{k}_p^{c,\mathrm{ad}}(\mathcal{U}) = \overline{\bigcup_{\widetilde{p}>p} I_{\widetilde{p},p}(\widetilde{\mathsf{k}}_{\widetilde{p}}^{c,\mathrm{ad}}(\mathcal{U}))}^{\|\cdot\|_{\widetilde{\mathsf{k}}_p^{c,\mathrm{ad}}(\mathcal{U})}}.$$

Then $V : K_p^{c,\mathrm{cond}}(\mathcal{U}) \to \mathsf{k}_p^{c,\mathrm{ad}}(\mathcal{U})$ is an isometric isomorphism. Moreover, the space $\mathsf{k}_p^{c,\mathrm{ad}}(\mathcal{U})$ is complemented in $\mathsf{k}_p^c(\mathcal{U})$ for $1 < p < \infty$ via the Stein projection. Since we may adapt Corollary 3.2.18 to $\mathsf{k}_p^c(\mathcal{U})$, we have

$$\mathsf{k}_p^c(\mathcal{U}) = [\mathsf{k}_{p_1}^c(\mathcal{U}), \mathsf{k}_{p_2}^c(\mathcal{U})]_\theta$$

holds isometrically for $1 \le p_1 < p_2 \le \infty$. This proves the Proposition for $1 < p_1 < p_2 < \infty$. Now we want to allow $p_1 \to 1$ and as in the proof of Proposition 3.7.2 we may assume $p_2 < 2$. By Theorem 3.6.24, we also have an algebraic atomic characterization $K_p^{c,\mathrm{cond}}(\mathcal{U}) = K_p^{c,\mathrm{cond,algat}}(\mathcal{U})$. Then we can adapt the first part of the proof of Proposition 3.7.2 to this case and obtain the inclusion

$$K_p^{c,\mathrm{cond}}(\mathcal{U}) \subset [K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})]_\theta.$$

Here we need the following result

$$K_q^{c,\mathrm{ad}}(\mathcal{U}) = [K_{q_1}^{c,\mathrm{ad}}(\mathcal{U}), K_{q_2}^{c,\mathrm{ad}}(\mathcal{U})]_\theta,$$

for $2 \le q_1 < q < q_2 < \infty$ such that $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. This comes from the fact that $K_q^{c,\mathrm{ad}}(\mathcal{U})$ is complemented in $K_q^c(\mathcal{U})$ via the Stein projection and Corollary 3.2.18 (ii). For the reverse inclusion we will show the dual version

$$(K_p^{c,\mathrm{cond}}(\mathcal{U}))^* \subset ([K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})]_\theta)^*. \tag{3.7.5}$$

Let $\varphi \in (K_p^{c,\mathrm{cond}}(\mathcal{U}))^*$ be a functional of norm < 1 and $\xi \in [K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})]_\theta$. Fix $\varepsilon > 0$. Then there exists an analytic function $f = (f_\sigma)^\bullet \in \mathcal{F}(K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U}))$ of norm

$$\|f\|_{\mathcal{F}(K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}),K_{p_2}^{c,\mathrm{cond}}(\mathcal{U}))} = \max\big(\sup_t \|f(it)\|_{K_{p_1}^{c,\mathrm{cond}}(\mathcal{U})}\ ,\ \sup_t \|f(1+it)\|_{K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})}\big) \leq \|\xi\|_\theta + \varepsilon$$

such that $f(\theta) = \xi$. On the other hand, recall that $K_p^{c,\mathrm{cond}}(\mathcal{U}) \subset \widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ isometrically, and since $p > 1$ by reflexivity of $\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U})$ we have

$$(\widetilde{K}_p^{c,\mathrm{cond}}(\mathcal{U}))^* = \Big(\prod\nolimits_{\mathcal{U}} L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))\Big)^* = \prod\nolimits_{\mathcal{U}} (L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*.$$

Thus by the Hahn-Banach Theorem there exists $\eta = (\eta_\sigma)^\bullet \in \prod_{\mathcal{U}} (L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*$ such that

$$\varphi(\xi) = \lim_{\sigma,\mathcal{U}} \tau(\eta_\sigma^* \xi_\sigma), \quad \text{for all } \xi = (\xi_\sigma)^\bullet \in K_p^{c,\mathrm{cond}}(\mathcal{U}).$$

Proposition 3.7.2 gives

$$(L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^* = ([L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))]_\theta)^*.$$

Note that since $L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))$ is reflexive, by [2, Corollary 4.5.2] we have

$$([L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)), L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))]_\theta)^* = [(L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*, (L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*]_\theta.$$

Hence for each σ, there exists an analytic function $g_\sigma \in \mathcal{F}((L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*, (L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*)$ of norm

$$\|g_\sigma\|_{\mathcal{F}((L_{p_1}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*,(L_{p_2}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*)} \leq C(p)\|\eta_\sigma\|_{(L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*}$$

such that $g_\sigma(\theta) = \eta_\sigma$. Setting

$$h(z) = \lim_{\sigma,\mathcal{U}} \tau(g_\sigma(\overline{z})^* f_\sigma(z)) \quad \text{for} \ \ z \in S$$

we get a continuous function on the strip S, analytic on the interior of S, satisfying the following estimates for $t \in \mathbb{R}$ and $k = 1, 2$

$$\begin{aligned} |h(k+it)| &\leq \lim_{\sigma,\mathcal{U}} \|g_\sigma(k+it)\|_{(L_{p_k}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*} \lim_{\sigma,\mathcal{U}} \|f_\sigma(k+it)\|_{L_{p_k'}^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma))} \\ &\leq C(p)\|\eta\|_{\prod_{\mathcal{U}}(L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c(\sigma)))^*}(\|\xi\|_\theta + \varepsilon). \end{aligned}$$

Thus the three-lines Theorem implies that

$$|\varphi(\xi)| = |h(\theta)| \leq C'(p)(\|\xi\|_\theta + \varepsilon).$$

Sending ε to 0 we obtain that $\varphi \in ([K_{p_1}^{c,\mathrm{cond}}(\mathcal{U}), K_{p_2}^{c,\mathrm{cond}}(\mathcal{U})]_\theta)^*$. This proves (3.7.5) and ends the proof. $\square$

3.7.3 Interpolation of Hardy spaces

Note that combining Proposition 3.6.17 with Proposition 3.2.38 we get

Proposition 3.7.9. *Let* $1 \leq p < 2$. *Then* $\mathcal{H}_p^c$ *is complemented in* $K_p^{c,\mathrm{cond}}(\mathcal{U})$.

Hence we deduce from Proposition 3.7.8 the analoguous interpolation result for the Hardy spaces $\mathcal{H}_p^c$.

Corollary 3.7.10. *Let* $1 \leq p_1 < p_2 < \infty$, $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$\mathcal{H}_p^c = [\mathcal{H}_{p_1}^c, \mathcal{H}_{p_2}^c]_\theta \quad \textit{with equivalent norms.}$$

Proof. The case $1 \leq p_1 < p_2 < 2$ is a direct consequence of Proposition 3.7.8 by complementation. We get the general case by using Corollary 3.2.43, by a standard application of the Wolff interpolation Theorem (see [51]). □

Standard duality arguments yield

Theorem 3.7.11. *Let* $1 < p < \infty$. *Then*

$$\mathcal{H}_p^c = [\mathcal{BMO}^c, \mathcal{H}_1^c]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

Proof. Applying the duality Theorem 4.5.1 of [2] to $\mathcal{H}_p^c = [\mathcal{H}_1^c, \mathcal{H}_q^c]_\theta$ where $\frac{1-\theta}{1} + \frac{1}{q} = \frac{1}{p}$, we get $\mathcal{H}_{p'}^c = [\mathcal{BMO}^c, \mathcal{H}_{q'}^c]_\theta$. Here we used Theorem 3.2.39 and Theorem 3.2.46. We conclude by an application of the Wolff interpolation Theorem, by using Corollary 3.7.10. □

This approach also works for the conditioned Hardy spaces h_p^c, by using $K_p^{c,\mathrm{cond}-}(\mathcal{U})$. Hence we have

Theorem 3.7.12. *Let* $1 < p < \infty$. *Then*

$$\mathsf{h}_p^c = [\mathsf{bmo}^c, \mathsf{h}_1^c]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

Remark 3.7.13. Note that for the interpolation of the conditioned Hardy spaces h_p^c we may also use the same approach as in Chapter 2, recalled in subsection 3.7.1. Indeed we may extend the interpolation result of Corollary 3.3.32 to the case $p = 1$ by using the continuous version of the Herz quasi-norm. Taking the limit in σ in (3.7.3), by Lemma 3.3.11 we can write the h_p^c-norm as an infimum as follows

$$\|x\|_{\mathsf{h}_p^c} \simeq \inf_\sigma N_{p,\sigma}^c(x) = \inf_\sigma \inf_{W_\sigma} \Big[\tau\Big(\sum_{t\in\sigma} w_t^{1-2/p}|d_{t^+}(x)|^2\Big)\Big]^{1/2} \quad \text{for } x \in L_2(\mathcal{M}).$$

Then, following the same steps as in Chapter 2 we obtain a different proof of Theorem 3.7.12.

Adapting the proof of Proposition 3.7.8 by using algebraic $\mathsf{h}_p^{1_c}$-atoms and Theorem 3.7.6, we obtain the following interpolation result for the diagonal spaces $\mathsf{h}_p^{1_c}$.

Theorem 3.7.14. *Let* $1 \leq p_1 < p_2 < 2$, $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. *Then*

$$\mathsf{h}_p^{1_c} = [\mathsf{h}_{p_1}^{1_c}, \mathsf{h}_{p_2}^{1_c}]_\theta \quad \textit{with equivalent norms.}$$

Remark 3.7.15. Observe that the Davis decomposition proved in Theorem 3.4.20 yields a different proof of the inclusion $\mathcal{H}_p^c \subset [\mathcal{H}_1^c, \mathcal{H}_q^c]_\theta$ for $0 < \theta < 1$, $1 < q < 2$ and $\frac{1-\theta}{1} + \frac{1}{q} = \frac{1}{p}$. Indeed, using Theorem 3.7.12 and Theorem 3.7.14 we obtain

$$\begin{aligned}\mathcal{H}_p^c &= \mathsf{h}_p^{1_c} + \mathsf{h}_p^d = [\mathsf{h}_1^{1_c}, \mathsf{h}_q^{1_c}]_\theta + [\mathsf{h}_1^c, \mathsf{h}_q^c]_\theta \\ &\subset [\mathsf{h}_1^{1_c} + \mathsf{h}_1^c, \mathsf{h}_q^{1_c} + \mathsf{h}_q^c]_\theta = [\mathcal{H}_1^c, \mathcal{H}_q^c]_\theta.\end{aligned}$$

We end this section with the interpolation result for the Hardy space $\mathcal{H}_p$.

Theorem 3.7.16. *Let* $1 < p < \infty$. *Then*

$$\mathcal{H}_p = [\mathcal{BMO}, \mathcal{H}_1]_{\frac{1}{p}} \quad \textit{with equivalent norms.}$$

Proof. We first show that

$$\mathcal{H}_p = [\mathcal{H}_1, \mathcal{H}_q]_\theta \quad \text{for } 1 < p < q < 2 \text{ and } \frac{1-\theta}{1} + \frac{1}{q} = \frac{1}{p}.$$

The direct inclusion comes from Corollary 3.7.10 by writing

$$\begin{aligned}\mathcal{H}_p &= \mathcal{H}_p^c + \mathcal{H}_p^r = [\mathcal{H}_1^c, \mathcal{H}_q^c]_\theta + [\mathcal{H}_1^r, \mathcal{H}_q^r]_\theta \\ &\subset [\mathcal{H}_1^c + \mathcal{H}_1^r, \mathcal{H}_q^c + \mathcal{H}_q^r]_\theta = [\mathcal{H}_1, \mathcal{H}_q]_\theta.\end{aligned}$$

For the reverse inclusion we use Theorem 3.2.56 and the fact that $\mathcal{H}_1$ embeds injectively into $L_1(\mathcal{M})$ by definition. Hence we get

$$[\mathcal{H}_1, \mathcal{H}_q]_\theta \subset [L_1(\mathcal{M}), L_q(\mathcal{M})]_\theta = L_p(\mathcal{M}) = \mathcal{H}_p.$$

Moreover, by Theorem 3.2.56 it is clear that for $1 < p_1 < p_2 < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ we have

$$\mathcal{H}_p = [\mathcal{H}_{p_1}, \mathcal{H}_{p_2}]_\theta.$$

We complete the proof by using the standard argument and Theorem 3.2.59. $\square$

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